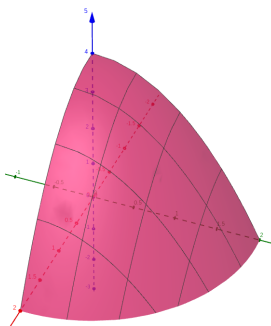1. (*15 Points*): Use cylindrical coordinates to set up **but do not evaluate**

$$\iiint_E x + y + z \ dV$$

where E is the solid in the first octant that lies under the paraboloid $z = 4 - x^2 - y^2$.

Solution:

$$\iiint_E x + y + z \ dV = \int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (r\cos(\theta) + r\sin(\theta) + z) \cdot r \ dz \ dr \ d\theta$$

2. (*15 Points*): Use spherical coordinates to set up **but do not evaluate**

$$\iiint_E y^2 \ dV$$

where E is the solid hemisphere $x^2 + y^2 + z^2 \leq 9$ with $y \geq 0$.

Solution:

$$\iiint_E y^2 \ dV = \int_0^{\pi} \int_0^{\pi} \int_0^3 (\rho\sin(\phi)\sin(\theta))^2 \rho^2 \sin(\phi) \ d\rho \ d\phi \ d\theta$$

$$= \int_0^{\pi} \int_0^{\pi} \int_0^3 \rho^4 \sin^3(\phi)\sin^2(\theta) \ d\rho \ d\phi \ d\theta$$

3. (*15 Points*): Find the Jacobian of the transformation, $x = u^2 + uv$, $y = uv^2$.

Solution:

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u + v & u \\ v^2 & 2uv \end{vmatrix} = 2uv(2u + v) - uv^2 = uv^2 + 4u^2v$$

4. (*15 Points*): Find the gradient vector field ∇f of $f(x, y) = y\sin(xy)$.

Solution:

$$\nabla f = \langle y^2 \cos(xy), xy\cos(xy) + \sin(xy) \rangle$$

5. (*20 Points*): Do one and only one of the following.

   (a) Evaluate
$$\int_C z \; dx + xy \; dy + y^2 \; dz$$
   where $C : x = \sin(t), y = \cos(t), z = \tan(t), -\pi/4 \le t \le \pi/4$.
   Solution:

$$\int_C z \; dx + xy \; dy + y^2 \; dz = \int_{-\pi/4}^{\pi/4} \tan(t)\cos(t) + \sin(t)\cos(t)(-\sin(t)) + \cos^2(t)\sec^2(t) \; dt$$

$$= \int_{-\pi/4}^{\pi/4} \sin(t) - \sin^2(t)\cos(t) + 1 \; dt$$

$$= -\cos(t) - \frac{1}{3}\sin^3(t) + t \; \Big|_{-\pi/4}^{\pi/4}$$

$$= -\frac{\sqrt{2}}{2} - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3 + \frac{\pi}{4} - \left(-\frac{\sqrt{2}}{2} - \frac{1}{3}\left(-\frac{\sqrt{2}}{2}\right)^3 - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2} - \frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)^3 = \frac{\pi}{2} - \frac{1}{3\sqrt{2}}$$

   (b) Use Green's Theorem to evaluate
$$\int_C y^3 \; dx - x^3 \; dy$$

   where C is the positively oriented circle $x^2 + y^2 = 4$.
   Solution:

$$\int_C y^3 \; dx - x^3 \; dy = \iint_D -3x^2 - 3y^2 \; dA$$

$$= -3 \int_0^{2\pi} \int_0^2 r^2 \; r \; dr \; d\theta$$

$$= -3 \int_0^{2\pi} \int_0^2 r^3 \; dr \; d\theta$$

$$= -6\pi \frac{r^4}{4} \Big|_0^2 = -24\pi$$

6. (*20 Points*): Do one and only one of the following.

   (a) Find a function f such that $\mathbf{F} = \nabla f$

   $$\mathbf{F} = \langle 2x\sin(yz) - y\sin(x) + 4, x^2 z\cos(yz) - 1/y + \cos(x), x^2 y\cos(yz) - 3z^2 \rangle$$

   Use it to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve $C : \mathbf{r}(t) = \langle 2t^2, t, e^t \rangle$ with $1 \le t \le 2$. Don't worry about simplifying the result.
   Solution: $f(x, y, z) = x^2\sin(yz) + y\cos(x) - \ln(y) - z^3 + 4x$ so

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = f(8, 2, e^2) - f(2, 1, e)$$
   $$= (8^2\sin(2e^2) + 2\cos(8) - \ln(2) - e^6 + 32) - (2^2\sin(e) + \cos(2) - \ln(1) - e^3 + 8)$$
   $$= 64\sin\left(2e^2\right) - 4\sin\left(e\right) + 2\cos\left(8\right) - \ln\left(2\right) - \cos\left(2\right) - e^6 + e^3 + 24$$
   $$\approx -310.2564047587718$$

   (b) Find the Curl and Divergence of the vector field

   $$\mathbf{F}(x, y, z) = zx^2\cos(y)\,\mathbf{i} + xy^3\sin(z)\,\mathbf{j} + yz^4 e^x\,\mathbf{k}$$

   Solution:

   $$\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zx^2\cos(y) & xy^3\sin(z) & yz^4 e^x \end{vmatrix}$$
   $$= (z^4 e^x - xy^3\cos(z))\,\mathbf{i} + (x^2\cos(y) - yz^4 e^x)\,\mathbf{j} + (y^3\sin(z) + zx^2\sin(y))\,\mathbf{k}$$
   $$\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$$
   $$= \frac{\partial}{\partial x}(zx^2\cos(y)) + \frac{\partial}{\partial y}(xy^3\sin(z)) + \frac{\partial}{\partial z}(yz^4 e^x)$$
   $$= 2zx\cos(y) + 3xy^2\sin(z) + 4yz^3 e^x$$

7. **Extra Credit:** (*10 Points*): Do either of the two exercises above that you did not choose for the previous problems. Do one and only one of these.