Notes on Infinity

First a quick review of some topics from discrete mathematics.

Definition 1: *A function or map* $f : X \to Y$ *is an* injection (one-to-one) *if whenever* $f(x_1) = f(x_2)$ *then* $x_1 = x_2$.

Definition 2: *A function or map* $f : X \to Y$ *is a* surjection (onto) *if for every* $y \in Y$ *there exist at least one* $x \in X$ *such that* $f(x) = y$.

Definition 3: *A function or map* $f : X \to Y$ *is a* bijection (a one-to-one correspondence) *if* f *is both injective and surjective.*

Theorem 1: *A bijective function* $f : X \to Y$ *has an inverse function, denoted* $f^{-1} : Y \to X$*, which is also a bijection.*

PROOF: Consult any text on discrete mathematics or logic, for example [1].

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Example 1: Consider the two sets $\mathbb{N}$ and $E = \{2n \mid n \in \mathbb{N}\}$ and the function $f : \mathbb{N} \to E$ defined as $f(x) = 2x$. The function f is a bijection. To show that the function is injective we assume that there are two elements, x_1 and x_2, in the domain, $\mathbb{N}$, such that $f(x_1) = f(x_2)$ and we proceed to prove that $x_1 = x_2$. In many cases this reduces simply to some algebra,

$$\begin{aligned} f(x_1) &= f(x_2) \\ 2x_1 &= 2x_2 \\ x_1 &= x_2 \end{aligned}$$

Hence the function f is an injection. To show that the function is a surjection we take an arbitrary element, y of the codomain E and find an element x in the domain $\mathbb{N}$ such that $f(x) = y$. If y is an arbitrary element of E we know, by the definition of E, that $y = 2x$ for some $x \in \mathbb{N}$. So if we take the number x as the desired element we have $f(x) = 2x = y$, proving that the function is surjective and therefore a bijection.

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Definition 4: *Two sets* A *and* B *have the same* cardinality *if there exists a bijective map* f *from* A *to* B.

Example 2: Consider the two sets $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Clearly they have the same size (or cardinality), 3. But by our definition we must show the existence of a bijection f between them. Easy enough, let $f : A \to B$ be defined as $f(a) = 1$, $f(b) = 2$ and $f(c) = 3$. This is clearly a bijection and hence our sets have the same cardinality.

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Notation: We denote the size, or cardinality of a set A by $|A|$. Note that if A is finite then $|A|$ is simply the number of elements in the set.

Example 3: Consider the two sets $\mathbb{N}$ and $E = \{2n \mid n \in \mathbb{N}\}$ from above and the function $f : \mathbb{N} \to E$ defined as $f(x) = 2x$. We have shown that this function is a bijection and hence E and $\mathbb{N}$ have the same cardinality, that is, they are the same size, $|\mathbb{N}| = |E|$. One nifty thing to note is that E is a proper subset of $\mathbb{N}$. So we have a proper subset that is the same size as the "bigger set". Some texts use this property to define an infinite set since this does not happen with finite sets. This also shows that the statement $A \subset B \Rightarrow |A| < |B|$ only applies to sets of finite size. Specifically, at least A being finite.

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In our last example we used the notation $|A| < |B|$. What does this notation really mean? If the sets A and B are finite it is obvious. If A is finite then $|A| = n$ for some $n \in \mathbb{N}$ and if B is finite then $|B| = m$ for some $m \in \mathbb{N}$ and $|A| < |B|$ simply means that $n < m$. What if the sets A and B are infinite? Does it make any sense to write $|A| < |B|$? How can we say $\infty < \infty$? This last inequality is nonsense, of course. The main problem with the last inequality is the symbol ∞. It implies that there is one and only one "type" or "size" of infinity. This can not be any further from the truth as we will soon see.

This still leaves us with the problem of what $|A| < |B|$ means if both A and B are infinite sets. Here is a definition,

Definition 5: *Given two sets A and B we will say that $|A| < |B|$ if there exists an injection $f : A \to B$ but there does not exist a bijection between the two. Similarly, $|A| = |B|$ means that there does exist a bijection between the two sets and finally, $|A| \leq |B|$ means that there is either an injection of A into B or there is a bijection between the two sets.*

Now we will take a quick look at different sizes of infinity.

Definition 6: *A set is said to be* countable *(or* enumerable *or* denumerable*) if there is a bijective map from it to a subset of $\mathbb{N}$. A set that is not countable is said to be* uncountable*.*

Hence any finite set is countable, since if A is finite then $|A| = n$ and we can clearly create a bijective map from A to the set $\{1, 2, 3, \ldots, n\}$. One thing to note is that some textbooks define a countable set as being any set that can be put into a one-to-one correspondence with $\mathbb{N}$. If this were our definition then clearly finite sets would not be countable. We could use either definition for what we need to do so there is no reason to split hairs on this topic. I personally think that saying a finite set is not countable goes against our intuition since we can clearly count a finite set. There may be times when we wish to restrict our attention to just infinite sets that are countable, in which case we will say that the set is *countably infinite*.

Now let's do a few more examples, some will be obvious but others may surprise you.

Example 4: The set of all integers $\mathbb{Z}$ is countable. To show this we simply need to establish a bijection between $\mathbb{Z}$ and $\mathbb{N}$. To do this we can simply write down several specific input/output values until our pattern is established or we can work toward some closed form formula for the entire map. One way to get this map is as follows. We will let $f : \mathbb{N} \to \mathbb{Z}$ be the map and define f as $f(0) = 0$, $f(1) = -1$, $f(2) = 1$, $f(3) = -2$, $f(4) = 2$, $\ldots$. For most audiences this is enough to establish the bijection. We could go one step further and write

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even.} \\ -(n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

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Our next example is a little surprising, at least I think so, and the map we will construct is a bit difficult to represent in closed form.

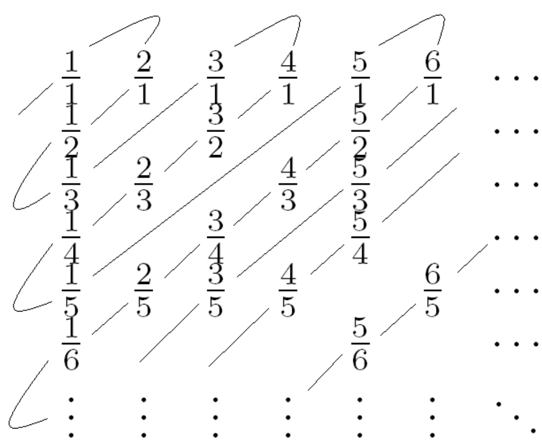
Example 5: The set of all rational numbers $\mathbb{Q}$ is countable. Recall that a rational number is any number that can be written in the form $\frac{a}{b}$ where $b \neq 0$. To show that this set is countable we will first show that the set of positive rational numbers is countable and then we can apply an argument similar to the one in the last example to get both the positive and negative rational numbers. So to show that the positive rational numbers are countable we need to establish a bijection from $\mathbb{N}$ to $\mathbb{Q}^+$. Here is the trick, write the set of rational numbers in a grid as follows.

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \cdots \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \frac{6}{5} & \cdots \\
\frac{1}{6} & \frac{2}{6} & \frac{3}{6} & \frac{4}{6} & \frac{5}{6} & \frac{6}{6} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Before we proceed we need to convince ourselves that this list actually contains all of the positive rational numbers. It should be fairly clear since if someone were to hand you the rational number $\frac{n}{m}$ (with both n and m positive) you could go to the m^{th} row and the n^{th} column and $\frac{n}{m}$ would be in that position. Now since a bijection is an injection we need to remove duplicates from the grid. Note that $\frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \cdots$, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \cdots$. So we will simply go through the grid and remove any number that has been previously encountered. Another way to interpret this action is removing any fraction that is not in lowest terms.

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} & \cdots \\
\frac{1}{2} & & \frac{3}{2} & & \frac{5}{2} & & \cdots \\
\frac{1}{3} & \frac{2}{3} & & \frac{4}{3} & \frac{5}{3} & & \cdots \\
\frac{1}{4} & & \frac{3}{4} & & \frac{5}{4} & & \cdots \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & & \frac{6}{5} & \cdots \\
\frac{1}{6} & & & & \frac{5}{6} & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Now to establish the map we zig-zag through the grid diagonally.



So our map would look something like this, $f(0) = 1$, $f(1) = 2$, $f(2) = \frac{1}{2}$, $f(3) = \frac{1}{3}$, $f(4) = 3$, $f(5) = 4$, $f(6) = \frac{3}{2}$, $f(7) = \frac{2}{3}$, $f(8) = \frac{1}{4}$, $\ldots$. To finish the example we apply the last example's method to get the negative numbers into the map. For ecample, $f(0) = 0$, $f(1) = -1$, $f(2) = 1$, $f(3) = -2$, $f(4) = 2$, $f(5) = -\frac{1}{2}$, $f(6) = \frac{1}{2}$, $f(7) = -\frac{1}{3}$, $f(8) = \frac{1}{3}$, $f(9) = -3$, $f(10) = 3$, $f(11) = -4$, $f(12) = 4$, $f(13) = -\frac{3}{2}$, $f(14) = \frac{3}{2}$, $f(15) = -\frac{2}{3}$, $f(16) = \frac{2}{3}$, $f(17) = -\frac{1}{4}$, $f(18) = \frac{1}{4}$, $\ldots$.

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This zig-zag method can be used to prove some other interesting facts about countable sets.

Theorem 2: *The union of a countable collection of countable sets is countable.*

Proof: Let A be the countable set of countable sets, so $A = \{A_1, A_2, A_3, \ldots\}$ with $A_1 = \{a_{1,1}, a_{1,2}, a_{1,3}, \ldots\}$, $A_2 = \{a_{2,1}, a_{2,2}, a_{2,3}, \ldots\}$, $A_3 = \{a_{3,1}, a_{3,2}, a_{3,3}, \ldots\}$, and so on. Since each of the A_i are countable and A is a countable collection of the A_i we can write the set of all elements of all the A_i into a single grid as,

$$\begin{array}{ccccc} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Now use the same zig-zag correspondence we used to show that the rational numbers were countable. In other words, order the union of elements as $\{a_{1,1}, a_{1,2}, a_{2,1}, a_{3,1}, a_{2,2}, a_{1,3}, a_{1,4}, a_{2,3}, a_{3,2}, a_{4,1}, \ldots\}$, and use this ordering as the bijection with $\mathbb{N}$.

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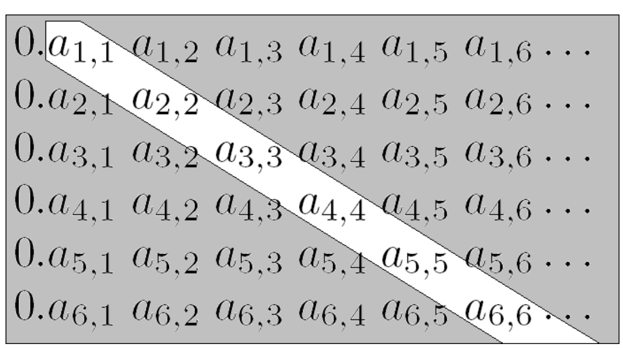
Now let's look at an example of a set that is not countable.

Example 6: The set of real numbers, $\mathbb{R}$, is uncountable. The proof we will use is called Cantor's Diagonalization Proof. It is a proof by contradiction. Before starting the proof recall that we can define the set of real numbers as the set of all decimal expansions. What we will show is that the set of real numbers between 0 and 1 is uncountable. That is, the set of numbers in the interval $[0, 1]$. If we establish that this set is uncountable then it clearly follows that the entire set of real numbers is uncountable. By way of contradiction we will assume that the set of real numbers in the interval $[0, 1]$ is countable. This implies that there is a one-to-one correspondence between $\mathbb{N}$ and $[0, 1]$. If there is then we can list all of the real numbers in $[0, 1]$ like so,

$$\begin{array}{l} 0.a_{1,1}\ a_{1,2}\ a_{1,3}\ a_{1,4}\ a_{1,5}\ a_{1,6} \cdots \\ 0.a_{2,1}\ a_{2,2}\ a_{2,3}\ a_{2,4}\ a_{2,5}\ a_{2,6} \cdots \\ 0.a_{3,1}\ a_{3,2}\ a_{3,3}\ a_{3,4}\ a_{3,5}\ a_{3,6} \cdots \\ 0.a_{4,1}\ a_{4,2}\ a_{4,3}\ a_{4,4}\ a_{4,5}\ a_{4,6} \cdots \\ 0.a_{5,1}\ a_{5,2}\ a_{5,3}\ a_{5,4}\ a_{5,5}\ a_{5,6} \cdots \\ 0.a_{6,1}\ a_{6,2}\ a_{6,3}\ a_{6,4}\ a_{6,5}\ a_{6,6} \cdots \\ \vdots \end{array}$$

The important thing to remember is that because of the assumption of countability we are guaranteed that *all* of the real numbers in $[0, 1]$ are in the above list. So to show that this set is uncountable we will simply find a real number in $[0, 1]$ that is not in this list. As for the notation in the above list, $a_{i,j}$ is the j^{th} decimal place of the i^{th} number. Now consider the diagonal of this grid, that is, the $a_{i,i}$ entries.



For each i select a number $b_i \neq a_{i,i}$ and construct the number $B = 0.b_1 b_2 b_3 b_4 b_5 b_6 \ldots$. We claim that B is not in the list we first constructed. B can not be equal to the first number in the list since the first decimal place of B differs from the first decimal place of $0.a_{1,1}a_{1,2}a_{1,3}a_{1,4}a_{1,5}a_{1,6}\ldots$. B can not be equal to the second number in the list since the second decimal place of B differs from the second decimal place of $0.a_{2,1}a_{2,2}a_{2,3}a_{2,4}a_{2,5}a_{2,6}\ldots$. B can not be equal to the third number in the list since the third decimal place of B differs from the third decimal place of $0.a_{3,1}a_{3,2}a_{3,3}a_{3,4}a_{3,5}a_{3,6}\ldots$, and so on. Hence B is a real number in the interval $[0,1]$ that is not in our original list. Hence we can not list all real numbers in the interval $[0,1]$ and therefore the set of real numbers in the interval $[0,1]$ is uncountable. As we mentioned above, this implies that the set of real numbers is uncountable.

If the above argument is too abstract consider this illustrative sub example, the list would look something like the following.

$$\begin{array}{l} 0.1248273994832\ldots \\ 0.2539188773628\ldots \\ 0.9923019928383\ldots \\ 0.1029938847722\ldots \\ 0.6472837462827\ldots \\ 0.5559828728393\ldots \\ 0.2847284950050\ldots \\ \vdots \end{array}$$

and the number B would be something like, $B = 0.4718952\ldots$

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Note what this says about irrational numbers like π, e, and $\sqrt{2}$. If we let I represent the set of irrational numbers we know that $\mathbb{R} = \mathbb{Q} \cup I$. If I were countable then $\mathbb{R}$ would be a union of two countable sets and hence be countable, which we know is not the case. Therefore the set of irrational numbers is uncountable. Thus there exists far more irrational numbers than rational numbers.

This also shows us that there are at least two different sizes of infinity, countable and uncountable. One can only ask is this the end of the story or is there more here. From our definition of cardinality it is clear that all countable sets have the same cardinality but what about the uncountable sets. All we know is that there is not a bijection between an uncountable set and $\mathbb{N}$, but that is all we know.

Recall from discrete mathematics the concept of the power set of a set. The power set of a set, A, is the set of all subsets of A.

Example 7: Let $A = \{a, b, c\}$ then the power set of A is

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$$

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As you know from discrete mathematics if A is finite, $|A| = n$ then the order of the power set of A is $|\mathcal{P}(A)| = 2^n$. So for finite sets it is obvious that A and $\mathcal{P}(A)$ have different cardinalities. Does the same hold for infinite sets? This question is a little harder to answer since both A and $\mathcal{P}(A)$ are infinite. To verify a difference in cardinalities we need to show that there does not exist a bijection between the two sets.

Theorem 3: $|A| < |\mathcal{P}(A)|$

Proof: Since there is an injection $g : A \to \mathcal{P}(A)$ defined by $g(a) = \{a\}$ we have $|A| \leq |\mathcal{P}(A)|$. By way of contradiction assume that $|A| = |\mathcal{P}(A)|$, then there exists a bijection $f : A \to \mathcal{P}(A)$. Consider the set $A' = \{a \in A \mid a \notin f(a)\}$. Since $A' \in \mathcal{P}(A)$ there exists $a' \in A$ with $f(a') = A'$. Now either $a' \in A'$ or $a' \notin A'$. If $a' \in A'$ then by the definition of A', $a' \notin f(a') = A'$, a contradiction. If $a' \notin A'$ then again by the definition of A', $a' \in f(a') = A'$, a contradiction. Thus no bijection f exists and $|A| < |\mathcal{P}(A)|$.

$\square$

This gives us an infinite sequence of progressively larger sizes of infinity.

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \cdots$$

Note that each of the sets $\mathcal{P}(\mathbb{N}), \mathcal{P}(\mathcal{P}(\mathbb{N})), \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}))), \ldots$ is uncountable. So there are an infinite number if sizes of infinity that are all uncountable.

One obvious question that arises from the above list is whether or not this list gets all of the sizes of infinity. That really breaks into two separate questions. First, is there a size of infinity that is larger than any of those in the above list? And is there a size of infinity that is between any of these?

The answer to the first questions is, yes. There are sizes of infinity larger than any that can be formed by $\mathcal{P}(\mathcal{P}(\cdots\mathcal{P}(\mathbb{N})\cdots))$. The answer to the second question is still unknown. To simplify the discussion we will introduce some new notation.

Notation: We denote $|\mathbb{N}| = \aleph_0$ and $|\mathbb{R}| = \aleph$. These numbers are referred to as infinite cardinals or infinite cardinal numbers.

Definition 7: *We define the sequence of infinite cardinals $\aleph_0$, $\aleph_1$, $\aleph_2$, $\ldots$ to mean that $\aleph_i < \aleph_{i+1}$ for all i and that there is no infinite cardinal $\aleph'$ such that $\aleph_i < \aleph' < \aleph_{i+1}$ for any i. In terms of sets this means that for any i there exists a set A_i such that $|A_i| = \aleph_i$. Also, there exists an injection from A_i to A_{i+1} for all i and there does not exist a bijection from A_i to A_{i+1}. Furthermore, it also means that there does not exist a set B such that there are injections from A_i to B and B to A_{i+1} and no bijections between A_i and B or B and A_{i+1}.*

Notation: Some logic texts will use exponential notation to denote the infinite cardinals. For example, $|\mathbb{N}| = \aleph_0$, $|\mathcal{P}(\mathbb{N})| = 2^{\aleph_0}$, $|\mathcal{P}(\mathcal{P}(\mathbb{N}))| = 2^{2^{\aleph_0}}$, $|\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| = 2^{2^{2^{\aleph_0}}}$, $\ldots$. In general, if a set A has size $|A| = p$, where p could be finite or infinite then the size of its power set is denoted $|\mathcal{P}(A)| = 2^p$.

Now we know that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \cdots$ and $|\mathbb{N}| = \aleph_0$. We also know, although we have not proven it, that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = \aleph$. One of our questions above can be stated as, does $\aleph_1 = \aleph$? That is, is $|\mathbb{R}|$ the next infinite cardinal? This question is still unknown but from the work that has been done so far it looks promising that this is indeed the case. This is known as the continuum hypothesis, note it is a hypothesis not a theorem since it has not been proven. Specifically, the continuum hypothesis states that $|\mathbb{R}| = \aleph_1$. That is, $|\mathcal{P}(\mathbb{N})| = \aleph_1$, so there is no different size of infinity between $|\mathbb{N}| = \aleph_0$ and $|\mathcal{P}(\mathbb{N})| = \aleph_1$. The generalized continuum hypothesis states that $2^{\aleph_i} = \aleph_{i+1}$ for all i. This is equivalent to saying that the sequence of infinite cardinals $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, $\ldots$ is $|\mathbb{N}|$, $|\mathcal{P}(\mathbb{N})|$, $|\mathcal{P}(\mathcal{P}(\mathbb{N}))|$, $|\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))|$, $\ldots$. So if we assume that the generalized continuum hypothesis is true then our sequence

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \cdots$$

has no sizes of infinity in between those displayed in the sequence. Furthermore we have examples of specific sets that have sizes $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, $\ldots$.

This still does not answer the question of whether or not $|\mathbb{N}|$, $|\mathcal{P}(\mathbb{N})|$, $|\mathcal{P}(\mathcal{P}(\mathbb{N}))|$, $|\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))|$, $\ldots$ is all of them. There is still a chance that there exists a set whose size is larger than all of these. That is, there could exist a set B in which there is an injection from A_i to B for all i and yet no bijection between B and any of the A_i. Our next goal is to prove that there is such a set B. The proof of the existence of such a set relies on the Axion of Choice. Notice that this is an axiom, not a theorem. That means that we either accept it or we do not. Most mathematicians accept the Axiom of Choice without hesitation but there are some who refuse to. We will, of course, accept it. Here are some equivalent statements of the axiom of choice. These and other equivalent statements can be found in [1] and [2].

Axiom of Choice 1: If $\mathcal{A}$ is a disjoint collection of nonempty sets, then there exists a set B such that for each A in $\mathcal{A}$, $B \cap A$ is a unit set, that is, contains a single element.

Axiom of Choice 2: For every set X there exists a function f on the collection, $\mathcal{P}(X) - \{\emptyset\}$, of nonempty subsets of X such that $f(A) \in A$.

The function f in the above statement of the axiom of choice is sometimes called a choice function because it reaches into a set and chooses an element from that set.

Axiom of Choice 3: If $\{A_i\}$ is a family of nonempty sets indexed by a nonempty set I, then $\mathrm{X}_{i\in I} A_i$ is nonempty. Where $\mathrm{X}_{i\in I} A_i$ represents the (possibly infinite) cartesian product of the A_i.

Now the existence of arbitrarily large cardinal numbers.

Theorem 4: *If $\mathcal{C}$ is a set of cardinal numbers, then there exists a cardinal number greater then each cardinal in $\mathcal{C}$.*

Proof: Since each cardinal number is associated with a set of that size we can consider $\mathcal{C}$ to be a disjoint collection of these sets. Using the axiom of choice we can define a representative set of the form $\mathcal{A} = \{A_u \mid u \in \mathcal{C}\}$, where $|A_u| = u$. Clearly, $|\cup_{u\in\mathcal{C}} A_u| \geq u$ for each $u \in \mathcal{C}$. Hence $|\mathcal{P}(\cup_{u\in\mathcal{C}} A_u)| = 2^{|\cup_{u\in\mathcal{C}} A_u|} > |\cup_{u\in\mathcal{C}} A_u|$, and so $|\mathcal{P}(\cup_{u\in\mathcal{C}} A_u)|$ is a cardinal number that exceeds all of the cardinal numbers in $\mathcal{C}$.

□

So the upshot is that $|\mathbb{N}|$, $|\mathcal{P}(\mathbb{N})|$, $|\mathcal{P}(\mathcal{P}(\mathbb{N}))|$, $|\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))|$, $\ldots$ is not all of them. We have,

$$\begin{aligned} &|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \cdots < |\mathcal{P}(\cup_{u\in\mathcal{C}} A_u)| < |\mathcal{P}(\mathcal{P}(\cup_{u\in\mathcal{C}} A_u))| < \cdots \\ &\quad < p < 2^p < 2^{2^p} < \cdots < q < 2^q < 2^{2^q} < \cdots \end{aligned}$$

where p is some cardinal larger than $|\cup\mathcal{P}(\cdots\mathcal{P}(\cup_{u\in\mathcal{C}} A_u)\cdots)|$ for example $|\mathcal{P}(\cup\mathcal{P}(\cdots\mathcal{P}(\cup_{u\in\mathcal{C}} A_u)\cdots))|$ and q is some cardinal larger than $|A_p \cup A_{2^p} \cup A_{2^{2^p}} \cup \cdots|$ for example $|\mathcal{P}(A_p \cup A_{2^p} \cup A_{2^{2^p}} \cup \cdots)|$. And this is not by any means an exhaustive list, at least as far as we know.

References

[1] *Set Theory and Logic*, by Robert R. Stoll, W. H. Freeman and Company, 1963.

[2] *The Foundations of Mathematics*, by Raymond L. Wilder, John Wiley & Sons, 1952.