Bayesian Modeling of Overdispersed Data with the Tilted Beta Binomial Distribution

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Binary data with $x$ successes and $n - x$ failures in $n$ trials is common and important category of data. Often modeled with the binomial distribution, $p(x) = \binom{n}{x} p^x (1 - p)^{n-x}$.

Binomial may be inadequate for real-world data because of overdispersion.

Beta binomial distribution is often used for such cases.

Unconditional modeling of data with beta binomial distribution dates back to Skellam (1948).
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Background

- Beta rectangular distribution proposed by Hahn (2008) for project management.
- Overdispersed alternative to Beta.
- Tilted beta distribution proposed by Hahn & López Martín (2015).
- Permits excesses at the extremes.

\[
p(p|\alpha, \beta, v, \theta) = (1-\theta)(2-2v+(4v-2)p)+\theta \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}
\]

Tilted beta also has \( p \) parameter. Accordingly, tilted beta binomial proposed by Hahn (2012).
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Figure: Examples of Tilted Beta Distributions for: $\alpha = 2, \beta = 3, v = 0$ (A); $\alpha = 2, \beta = 3, v = 0.5$ (B), $\alpha = 2, \beta = 3, v = 1$ (C), $\alpha = 3, \beta = 2, v = 0$ (D); $\alpha = 3, \beta = 2, v = 0.5$ (E), $\alpha = 3, \beta = 2, v = 1$ (F)
Tilted Beta Binomial Distribution

Compounding the tilted beta with the binomial produces the distribution.

\[ p(y|\alpha, \beta, \theta, v) = (1 - \theta) \frac{2(y(2v - 1) + n(1 - v) + 1)}{(n + 1)(n + 2)} + \theta \binom{n}{y} \frac{\Gamma(\alpha + \beta)\Gamma(\alpha + y)\Gamma(\beta + n - y)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha + \beta + n)}. \]  

(2)
Tilted Beta Binomial Distribution

Figure: Examples of Tilted Beta Binomial Distributions

\[ \nu = 0.3, \alpha = 3, \beta = 4 \]

Both plots: ○, \( \theta = 0.25 \); □, \( \theta = 0.5 \); ◊, \( \theta = 0.75 \).
CDF for Tilting Binomial

- The tilting mixture component, preceded by $1 - \theta$ in (2), can be called tilting binomial.
- Unlike beta binomial, this distribution has a relatively simple form for CDF.

$$F(y|v, n) = \begin{cases} 
1 & \text{if } y = n, \\
\frac{y(2v-1)+2n(1-v)+2}{(n+2)(n+1)}(y+1) & \text{if } 0 \leq y \leq n, \\
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Several ways to estimate a compounded mixture using Bayesian methods.

Approach A: Use (2) as *marginalized* likelihood, place priors on $\alpha, \beta, \theta, \nu$.

Approach B: Use binomial distribution as *conditional* likelihood, give $p$ a tilted beta ‘quasi-prior’, place ‘real’ priors on $\alpha, \beta, \theta, \nu$.

Approach C: Use the approach familiar from Dempster, Laird & Rubin (1977). For a two component mixture with densities $f_1(\cdot)$ and $f_2(\cdot)$, form the likelihood as $f_1(\cdot)^z_i f_2(\cdot)^{1-z_i}$ where $z_i$ is a latent binary variable with Bernoulli parameter $\theta$.

Approach D: Combination of Approach B and C.
Tilted Beta Binominal Distribution

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Approach D: Combination of Approach B and C.
Data for state of Maryland from the 2010 U.S. Census. There are 518 Census places in Maryland. $x$ is number of people indicating African-American and of one race.

- Bayesian MCMC.
- 5000 MCMC burn-in iterations and 200,000 estimation. WinBUGS.
- Use transformations $\alpha = \mu \phi$ and $\beta = \phi(1 - \mu)$ for beta component parameters.
- Uniform $(0,1)$ priors on $\mu$, $\theta$, and $\nu$. Uniform $(0,100)$ prior on $\phi$.
- We examine Monte Carlo (MC) error and estimation time.
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Modeling

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## Tilted Beta Binomial Results

### Table: Parameter Estimates: Posterior Means

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Approach A</th>
<th>Approach B</th>
<th>Approach C</th>
<th>Approach D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.5437</td>
<td>0.5439</td>
<td>0.5439</td>
<td>0.5437</td>
</tr>
<tr>
<td>$\beta$</td>
<td>5.911</td>
<td>5.925</td>
<td>5.925</td>
<td>5.906</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.08674</td>
<td>0.08668</td>
<td>0.08668</td>
<td>0.08684</td>
</tr>
<tr>
<td>$\phi$</td>
<td>6.454</td>
<td>6.469</td>
<td>6.469</td>
<td>6.45</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.6478</td>
<td>0.647</td>
<td>0.647</td>
<td>0.6484</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.7461</td>
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Approaches agree with regard to posterior means.
As mentioned, 200,000 iterations.

Approach B (Binomial) was the fastest, 40% less time than Approach A.

Approach C (‘EM’) took approximately 40% more time than Approach A.

Approach D (‘EM’) took approximately 17% more time than Approach B.

Conclusion: Binomial likelihood approach is faster for WinBUGS.
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‘Digestible’ comparison of MC errors possibly difficult.

However, note that $\mu$, $\theta$ and $\nu$ are on common $(0, 1)$ range.

We average the ratios of MC errors for these parameters.

Approach C (‘EM’) had about a 52% greater MC error than Approach A.

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- ‘EM’ approaches increase MC error. Marginalizing reduces MC error.
- However, using conditional approach only increases MC error modestly vs. marginalized.

Summary: multiple approaches to estimation of tilted beta binomial models exist and data analysis can proceed with different approaches based on goals.
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Selected References


