Abstract II
Section 21
extras

#1

In the ring $M(\mathbb{C})$, let

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

The product of a real number and a matrix is the matrix given by this rule:

$$r\begin{pmatrix} t & u \\ v & w \end{pmatrix} = \begin{pmatrix} rt & ru \\ rv & rw \end{pmatrix}$$

The set H of **real quaternions** consists of all matrices of the form

$$\begin{aligned} a\mathbf{1} + b\mathbf{j} + c\mathbf{j} + d\mathbf{k} &= a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} bi & 0 \\ 0 & -bi \end{pmatrix} + \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} + \begin{pmatrix} 0 & di \\ di & 0 \end{pmatrix} \\ &= \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}, \end{aligned}$$

where a, b, c, and d are real numbers.

You may assume:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1} \qquad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}$$
$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i} \qquad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

- Show that H is a noncommutative ring with identity.
- Show that H is a division ring (~~defined in Exercise 34~~). [*Hint:* If $M = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, then verify that the solution of the equation $Mx = \mathbf{1}$ is the matrix $ta\mathbf{1} - tb\mathbf{i} - tc\mathbf{j} - td\mathbf{k}$, where $t = 1/(a^2 + b^2 + c^2 + d^2)$.]
- Show that the equation $x^2 = -1$ has infinitely many solutions in H. [*Hint:* Consider quaternions of the form $0\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where $b^2 + c^2 + d^2 = 1$.]

#2

Consider $\mathbb{R} \times \mathbb{R}$ with the usual coordinatewise addition and a new multiplication given by

$$(a,b)(c,d) = (ac - bd, ad + bc).$$

Carefully
Show that with these operations $\mathbb{R} \times \mathbb{R}$ is a field.