10.3 Problem 14

$$f(x) = \begin{cases} x & 0<x<\pi \\ x+\pi & -\pi<x<0 \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{\pi} x\,dx + \int_{-\pi}^{0}\pi\,dx\right\}$$

$$= \frac{1}{\pi}\left\{0 + \pi x\Big|_{-\pi}^{0}\right\} = \frac{1}{\pi}\left\{0-(-\pi^2)\right\}$$

$$= \pi$$

$n \geq 1$

$$a_n = \frac{1}{\pi}\left\{\underbrace{\int_{-\pi}^{\pi} x\cos nx\,dx}_{\text{odd} \;\Rightarrow\; 0} + \int_{-\pi}^{0}\pi\cos nx\,dx\right\} = \frac{1}{\pi}\left\{\frac{\pi}{n}\sin nx\Big|_{-\pi}^{0}\right\} = 0$$

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{\pi} x\sin nx\,dx + \int_{-\pi}^{0}\pi\sin nx\,dx\right\} = \frac{1}{\pi}\left\{-\frac{x}{n}\cos nx\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{1}{n}\cos nx\,dx - \frac{\pi}{n}\cos nx\Big|_{-\pi}^{0}\right\}$$

$u=x \quad v=-\frac{1}{n}\cos nx$

$du=dx \quad dv=\sin nx\,dx$

$$= \frac{1}{\pi}\left\{-\frac{\pi}{n}(-1)^n - \frac{\pi}{n}(-1)^n + 0 - \frac{\pi}{n}\left(\cos 0 - \cos(n\pi)\right)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{2\pi}{n}(-1)^{n+1} - \frac{\pi}{n}\left(1-(-1)^n\right)\right\}$$

if n is even

$$b_n = \frac{2\pi}{\pi n}\{-1\} = -\frac{2}{n} \Rightarrow b_{2n} = -\frac{1}{n} \qquad b_{2n+1} = 0$$

if n is odd

$$b_n = \frac{1}{\pi}\left\{\frac{2\pi}{n} - \frac{2\pi}{n}\right\} = 0$$

$$\boxed{f(x) \sim \frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{\sin 2nx}{n}}$$