0 since $\sin(n\pi)=0$.

Math 311 Test 3 Name KMS

FYI: $\displaystyle\int_{-\pi}^{\pi} x^2\cos(nx)dx = \left.\frac{2x\cos(nx)}{n^2}\right]_{-\pi}^{\pi} + \left\{\frac{(n^2x^2-2)\sin(nx)}{n^3}\right\}_{-\pi}^{\pi} + C$ and

$\displaystyle\int x^2\sin(nx)dx = \frac{2x\sin(nx)}{n^2} - \frac{(n^2x^2-2)\cos(nx)}{n^3} + C$

$y'' = 2xy' + 2y$

$y''(0) = 2(1)$

$y'''(x) = 2xy'' + 2y' + 2y' = 0$

$y''''(x) = $ etc ...

1. Find the first four non-zero terms in the power series solution for:

y"(x) - 2xy'(x) -2y(x) = 0; y(0)=1; y'(0) = 0.

$a_0 = 1 \quad a_1 = 0$

$$a_{k+2} = \frac{2(k+1)a_k}{(k+2)(k+1)} = \frac{2a_k}{k+2}$$

$$\sum_{n=2}^{\infty}{}' n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty}{}' 2na_nx^n - \sum_{n=0}^{\infty}{}' 2a_nx^n = 0$$

$$\sum_{k=0}^{\infty}{}'(k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty}{}' 2ka_kx^k - \sum_{k=0}^{\infty}{}' 2a_kx^k = 0$$

$$2a_2 - 2a_0 + \sum_{k=1}^{\infty}{}'\{(k+2)(k+1)a_{k+2} - 2(k+1)a_k\}x^k = 0$$

$a_2 = a_0 = 1$

$a_3 = \frac{2(0)}{3} = 0 = a_5 = \dots = a_{2n+1}$

$a_4 = \frac{2a_2}{4} = \frac{1}{2}$

$a_6 = \frac{2a_4}{6} = \frac{1}{6}$

$a_8 = \frac{2a_6}{8} = \frac{1}{24}$

$y \sim 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24}$

$a_{2n} = \frac{1}{n!}$

$y = \sum_{n=0}^{\infty}{}' \frac{x^{2n}}{n!} = e^{x^2}$.

2. Find the Fourier series on [-π,π] for f(x) = x².

f is even so $b_n = 0\ \forall n$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2dx = \frac{2}{\pi}\int_0^{\pi} x^2dx = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$

$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \frac{2}{\pi}\left\{\frac{2x\cos(nx)}{n^2}\right\}_0^{\pi} = \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right] = \frac{4(-1)^n}{n^2}$

$$f(x) \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty}{}' \frac{(-1)^n4}{n^2}\cos nx$$

3. If u(x,t) is a solution to $\dfrac{\partial^2 u}{\partial t^2} = \beta\dfrac{\partial u}{\partial x}$ and u(x,t) = X(x)T(t), what ordinary differential equations do X and T satisfy?

$XT'' = \beta X'T \Rightarrow \frac{T''}{T} = \frac{\beta X'}{X} = K$ or $\frac{T''}{\beta T} = \frac{X'}{X} = K$

$T'' = KT$ & $\beta X' = KX$ or $T'' - KT = 0$, $\beta X' - KX = 0$

alternatively $T'' = K\beta T$, $X' = KX$ $\Rightarrow$ $T'' - K\beta T = 0$, $X' - KX = 0$.

4. Find a solution to the following IBVP:

$\frac{\partial^2 u}{\partial t^2} = 25\frac{\partial^2 u}{\partial x^2}$ for 0<x<2; t>0

u(0,t)=u(2,t) = 0

u(x,0) = sin(πx) + 3 sin(7πx) -sin(10πx)

u_t(x,0) = 5π sin(πx) + 20 π sin(2πx)

$u(x,t) = \sum{}'\left[a_n\cos\frac{n\pi\alpha}{L}t + b_n\sin\frac{n\pi\alpha}{L}t\right]\sin\left(\frac{n\pi}{L}x\right)$

$B_n = \frac{n\pi\alpha}{L}b_n$ $\quad L=2 \quad \alpha=5 \quad b_n = \frac{L}{n\pi\alpha}B_n$

$b_2 = \frac{1}{5\pi}(5\pi) = 1 \quad b_4 = \frac{1}{2\pi5}(20\pi) = 2$

$u(x,t) = \cos(5\pi t)\sin(\pi x) + 3\cos(35\pi t)\sin(7\pi x) - \cos(50\pi t)\sin(10\pi x)$
$+ \sin(5\pi t)\sin(\pi x) + 2\sin(10\pi t)\sin(2\pi x)$

Extra Credit:

5. Find all the eigenvalues and associated eigenfunctions for the boundary value problem:

y"+λy=0 y'(0) = 0; y'(2π) = 0

if $\lambda<0$ then $y = c_1e^{\sqrt{\lambda}t} + c_2e^{-\sqrt{\lambda}t}$

$y' = \sqrt{\lambda}(c_1e^{\sqrt{\lambda}t} - c_2e^{-\sqrt{\lambda}t})$

$y'(0) = \sqrt{\lambda}(c_1 - c_2) = 0 \Rightarrow c_1 = c_2$

$y'(2\pi) = \sqrt{\lambda}c_1(e^{\sqrt{\lambda}(2\pi)} - e^{-\sqrt{\lambda}2\pi}) = 0$

if $e^{2\sqrt{\lambda}\pi} = e^{-2\sqrt{\lambda}\pi}$ th $\sqrt{\lambda} = -\sqrt{\lambda} = 0 \neq 0$ so NO NTS's

if $\lambda = 0$: $y = c_1t + c_2$, $y' = c_1 = 0$, $c_2 =$ anything, $y = c$ is NTS.

if $\lambda>0$: $y = c_1\sin\sqrt{\lambda}t + c_2\cos\sqrt{\lambda}t$

$y' = \sqrt{\lambda}(c_1\cos\sqrt{\lambda}t - c_2\sin\sqrt{\lambda}t)$

$y'(0) = \sqrt{\lambda}c_1 \Rightarrow c_1 = 0$

$y'(2\pi) = \sqrt{\lambda}(-c_2\sin(\sqrt{\lambda}2\pi)) = 0 \Rightarrow \sqrt{\lambda}2\pi = n\pi$

so $\sqrt{\lambda} = \frac{n}{2} \Rightarrow \lambda = \frac{n^2}{4}$

So Eigenvalues are $\lambda = n^2/4 \quad n = 0, 1, 2, \dots$

associated functions are $\cos\sqrt{\lambda}t = \cos\left(\frac{nt}{2}\right) \quad n = 0, 1, \dots$

6. Find the general term for the series solution to problem 1. Identify the function to which the series converges and verify that it is a solution to the DE. (See Above)