Developing Students' Understanding of Decimal Numeration in Grade 5

Lindsay Gordon, Felicia De Peña & Faculty Mentor
Dr. Claudia Burgess

Introduction:

Many misconceptions exist in the realm of decimal numeration (Martinie, 2014, University of Melbourne, 2012). Teachers and students alike struggle with the notions that:

- The lengths of decimals do not dictate the value (e.g. 0.65 is not greater than 0.7 even though 0.65 has more digits),
- Strategies such as money thinking and whole-number thinking do not dictate accuracy (e.g. students who think in terms of money are at times able to reason only to the hundredths place and not beyond)
- Familiarity with a standard algorithm does not dictate understanding (e.g. lining up decimal points and annexing zeroes when needed to add or subtract does not prompt students to think about the sizes of the numbers they are dealing with).

Conventional teaching strategies often fail to help students overcome these misconceptions. Stacey (2001) found that one and five pre-service teachers did not have a well-integrated knowledge of decimal numeration. Under-developed content and pedagogical knowledge often leads to teaching decimal concepts by rote. For instance, teachers often tell students to annex zeroes in order to compare decimals such as 0.6 and 0.65, which prevents them from developing true place value understanding (D'Ambrosio & Kastberg, 2012).

The purpose of this study was to examine four students' existing knowledge of decimal numeration and explore strategies for developing their mathematical proficiency in regard to decimals.

The research questions for this study were:

- What difficulties do the students have in reasoning about decimals and place value?
- What teaching strategies and representations can help the students accurately compare two decimal representations?
- What teaching strategies and representations can help the students learn decimal addition and subtraction and understanding.

References:
D'Ambrosio, B.S., & Kastberg, S.E. (2012). Building understanding of decimal fractions. *Teaching Children Mathematics, 18*, 559-564.
Martinie, S.L. (2014). Decimal fractions: An important point. *Mathematics Teaching in the Middle School, 19*(7), 420-429.
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Theoretical framework:

- We conceptualized *mathematical proficiency* using the National Research Council's (2001) five strands. These strands are:

Conceptual Understanding	Procedural Understanding	Strategic Understanding	Adaptive Reasoning	Productive Disposition
• Refers to a student's ability to learn by connecting ideas to their pre-existing schema. When students truly understand a concept, they know the foundational knowledge behind any procedure they may be carrying out.	• Defines a student's ability to carry out processes flexibly, accurately, efficiently and appropriately.	• Refers to a student's ability to discover and formulate their own techniques and represent them mathematically.	• Is the capacity for logical thought, reflection, explanation and justification. When students employ their adaptive reasoning, they are able to apply strategies in different contexts and articulate what they're doing and why they are doing it.	• Is the inclination to see mathematics as sensible, useful and worthwhile, as well as a student's own commitment and value of the math he/she is working on.

- The Common Core State Standards Writing Team (2012) suggested a learning progression for decimal numeration. Steps in the progression relevant to our research project include for grade four:
  - **Generalize place value understanding for multi-digit whole numbers-** In the base-ten system, the of each place is 10 times the value of the place to the immediate right.4.NBT.1 Because of this, multiplying by 10 yields a product in which each digit of the multiplicand is shifted one place to the left (p. 12).
- And for grade five:
  - **Students extend their understanding of the base-ten system to decimals to the thousandths place-** Multiplying by 10 once shifts every digit of the multiplicand one place to the left in the product (the product is ten times as large) because in the base-ten system the value of each place is 10 times the value of the place to its right (p. 16)
  - **Students use the same place value understanding for adding and subtracting decimals that they used for adding and subtracting whole numbers-** The graphic below shows how students learn to subtract whole numbers with re-grouping in second grade (p. 9). It can be applied to learning how to subtract decimal numerations.

Additional literature used to guide the research project included:

"Promoting Decimal Number Sense and Representational Fluency" (Suh et al., 2008):
- This article stresses the importance of representational fluency in building mathematical understanding. In the study Suh used decimal grids, place-value charts, symbols and vocabulary as manipulatives. Suh found that using multiple representations aided in making generalizations, encouraged flexibility, and aided the communication of mathematical ideas (including ELL students).

"Investigating Students' of Decimal Fractions" (Martinie & Bay-Williams, 2003):
- This article focuses on the importance of learning mathematics by building on prior knowledge. Since basing the knowledge of decimals on that of whole numbers can be misleading, the researchers presented students with multiple contexts and representations numerous times. They found that students improved when conceptual teaching methods were applied.

"Progression for the Common Core State Standards for Mathematics"
- In this progression, the writing team suggested students compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols <, =, or > , and justify the conclusions,
- e.g., by using a visual model (NF 3-5 p.10).

References
Common Core Standards Writing Team. (2013). *Progression for the common core state standards for mathematics (draft), number and operation- fractions, 3-5, measurement and data*. Retrieved from http://commoncoretools.me/wp-content/uploads/2011/08/ccss_progression_nf_35_2013_09_19.pdf
National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J Kilpatrick, J. Swafford, and B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.
Martinie, S., & Bay-Williams, J. (2003). Investigating students' of decimal fractions. *Mathematics Teacing in the Middle School, 8*(5), 244-247.
Suh, J. Johnson, C., Jamieson, S., & Mills, M. (2008). Promoting decimal number sense and representational fluency. *Mathematics Teaching in the Middle School*, 14(1), 44-50.

Methodology: Participants and procedure

In this study, four students getting ready to enter fifth grade were chosen from local elementary schools. The students, Jane, Kaylie, Adam, and Nick (pseudonyms) were brought in for a pre-test and interview before beginning the tutoring process. The tutoring sessions were held one hour a week for seven weeks and followed up by a post-test and interview. Three of the students attended all sessions and interviews. One student missed an instructional session but participated in all other activities.

Over the duration of the study, instruction was designed to help students attain the following Common Core State Standards (CCSS):

-Recognize that a particular digit is ten times smaller than the place on its left. (CCSS.Math.Content.5.NBT.A.1)
-Compare, read and write decimals to the thousandths place. (CCSS.Math.Content.5.NBT.A.3)
-Compare, read and write decimals using base-ten numerals, expanded form and number names. (CCSS.Math.Content.5.NBT.A.3.a)
-Compare decimals using >, =. < symbols to the thousandths place. (CCSS.Math.Content.5.NBT.A.3.b)
-Add and subtract decimals of different sizes using drawings and other manipulatives. (CCSS.Math.Content.5.NBT.B.7)

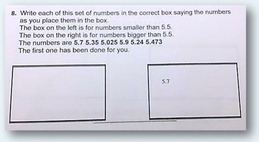
*National Governors Association for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Washington, DC: Author. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

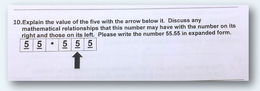
We used an instructional cycle which integrated teaching and research. We analyzed the students' work and classroom interaction each week to determine a set of goals for the following week's tutoring session. Using the classroom data and our data-based conjectures about how to develop student learning, we created to move the students' thinking forward along the strands of mathematical proficiency. This cycle repeated each week, as shown in the figure above.

Methodology: Data gathering and analysis

Prior to the tutoring sessions, each student was required to take a written assessment and an interview with the researchers – each assessment lasted approximately 30 minutes. The written assessment consisted of 25 multiple choice questions about decimals ranging from place value to the multiplication and division of decimals. During the interview students were asked a variety of questions regarding decimals. The interview was composed of 17 open-ended questions designed to use multiple operations. The researchers recorded the interview and asked questions to gauge the level of student understanding.


6. **Use decimals to solve the following problem.** Margie bought 8 apples that cost 47 cents each. She paid with a five-dollar bill. How much change did Margie receive?


8. Write each of this set of numbers in the correct box saying the numbers as you place them in the box.
The box on the left is for numbers smaller than 5.5.
The box on the right is for numbers bigger than 5.5.
The numbers are **5.7 5.35 5.025 5.9 5.24 5.473**
The first one has been done for you.
5.7


10.Explain the value of the five with the arrow below it. Discuss any mathematical relationships that this number may have with the number on its right and those on its left. Please write the number 55.55 in expanded form.
5 5 * 5 5 5

Researchers were given a video camera to record tutoring sessions and interviews. Camera(s) were set up in the classroom to record the students' verbal and physical work and their interactions with the researchers and other students. After each one hour session all written work was collected and the videos were transcribed verbatim. We collaboratively analyzed the sessions with our faculty mentor. We reviewed each lesson transcript and found the strengths and weaknesses students exhibited in each of the five strands of mathematical proficiency. We then completed an "instructional lesson summary", which summarized the data and aided in planning the following lesson.

References:
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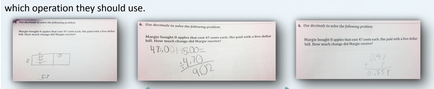
Georgia Department of Education. 2014. Retrieved from https://www.georgiastandards.org/Common-Core/Common%20Core%20Frameworks/CCGPS_Math_5_Unit2Framework.pdf
Performance Assessment Task, *Noyce Foundation. 2009. Retrieved from http://www.insidemathematics.org/assets/common-core-math-tasks/decimals.pdf*

Empirical Teaching and Learning Trajectory:

Initial Assessment Results

After the initial assessment and interview, it was striking that all four students were lacking very basic conceptual understanding of place value, even in reference to whole numbers. It was clear that we needed to go back and work on some Common Core State Standards from fourth grade before moving onto fifth. All four students struggled to explain what they knew about decimals, and could really only relate them to money.

When presented with problems including decimal numeration, they used algorithms and procedures they had learned in school. For example, when adding two decimals together, students would line up the decimal points, but when asked for further explanation, they could not articulate why they had done so. When presented with word problems, students struggled to determine which operation they should use.



As you can see, all three of these students used different incorrect operations to solve the same problem.

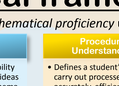
Students were also unable to translate numbers between representations (e.g. visual models, word models, numeral representations, etc.). All students struggled with strategic competence and adaptive reasoning. When presented with an unfamiliar problem, they often became frustrated. When asked why they were performing a task, they would lack confidence and would usually immediately change their answer.

Jane: Reese ran zero point five of a mile. Jen ran zero point forty-five of a mile. Reese thinks she ran more than Jen. Do you agree or disagree? Justify your answer with a written explanation.
LG: …You can do a visual model.
Jane: Ugh
LG: …Why would you say Reese? Just tell me why you thought that.
Jane: Because zero decimal five compared to this,
LG: Is it greater than or less than?
Jane: Less than.
LG: Which is less than?
Jane: This one
LG: Why? Don't like change your answer just because I'm asking you why. I'm just trying to follow what you're thinking about… What made you say that?…
Jane: Because if you compare 5 of the zero point five and zero point forty-five, this one would be greater.
LG: But why? Can you draw me a picture?
Jane: I don't think I know really how to do this yet.

In the initial assessment, this student was unsure about how to approach this problem. When suggested she use a visual model, she resists. When asked why she chose a particular answer, she struggles to explain herself and then changes her answer, assuming it is incorrect.

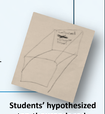
Instructional cluster 1

During the first two weeks of tutoring students participated in activities to solidify their knowledge of place value. In the first lesson, we focused on whole number place value. Students were introduced to the base ten blocks (shown on the right) and asked questions regarding their prior knowledge of the blocks.



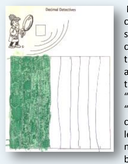
Students were then encouraged to find a pattern (cube, rod, flat) and hypothesis what block would come after a thousands' block using the pattern they discovered. The students then drew on a sheet of butcher paper what they believed to be the next piece (ten thousands' rod). In this lesson students also discovered that each place value is ten times the size of the one in front of it.

Kailey: All together it would be ten thousand, but for each box it would be two thousand.
LG: Right, but all together it would be-
Kailey: Ten thousand
LG: So do you think this would be our next piece? What do you think- Which piece does this look like?
Kailey: A rod.
LG: A rod, so do you see any patterns?
Nick: No
Adam: Ooh! I see it I see it!
LG: What do you see, Adam?
Adam: 1 times 10 equals 10, 10 times 10 equals 100, 100 times 10 equals 1000 and-
Nick: Ta da!



Students' hypothesized ten-thousands rod.

Lesson 2 focused on the place value of numbers to the right of the decimal point. For this lesson, students were provided with an activity sheet and a magnifying glass. The activity sheet had two squares, a 1x1 inch square and a 7x7 inch square. Students were asked to split the small box into 10 even pieces, then into 100 pieces. The students then used the magnifying glasses to look at the work they had done.

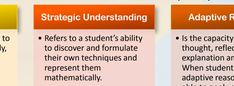


Student activity sheet.

Most students noted the difficulty of this activity. After completing the task, we asked the students to use the bigger square to represent 1 rather than the small one. Students then cut out tenths, hundredths and thousandths pieces and used them to represent numerals, giving names to the place values, and emphasizing the importance of the names of numbers. At this point, students started to distinguish between "tens" and "tenths", between "hundreds" and "hundredths," and "thousands" and "thousandths" by discussing sizes of pieces compared to one another. This prepared them for the following lesson, in which they reviewed decimal place values and the number names by using a large "Minecraft" themed board to build a wall made of tenths, hundredths and thousandths pieces.

Instructional cluster 2

Once students started showing a strong foundational concept of place value, we moved on to visual representations of decimals. We started doing this by having them use the following decimal grids:



These grids helped the students solidify their understanding of place value and provide a visual of the relationships between the two representations.

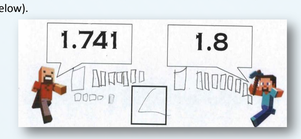
- Students became more articulate in their descriptions of the relationships between the decimal places. For example, students explained that they would rather have 6 tenths of a cake instead of 6 hundredths of a cake because the hundredths pieces would be ten times smaller than the tenths pieces.
- Students expressed understanding of equivalency. They understood that adding a zero doesn't change the value of the decimal numeration, although they are read differently. For example, 0.47 and 0.470 are read differently ("forty-seven hundredths" vs. "four hundred seventy thousandths"), but are equivalent because they are of the same value.
- Students discovered an efficient strategy to name decimals and justified it by explaining, "You name it by the smallest piece."

2.617
Two ones, six tenths, one hundredth and seven thousandths

2.617
Two and six hundred seventeen thousandths

- Through "Decimal Bingo," students became more fluent in translating between different representations of decimal numerations. These different representations included:
  - Numeral form (e.g. 8.71)
  - Word form, both written and spoken (e.g. Eight and seventy one hundredths)
  - Expanded form (e.g. (8x1)+(7x*110*)+(1x*1100*))

After becoming comfortable with the provided grids, students moved on to drawing their own representations of decimals. This proved to be useful when students were asked to compare two numbers and decide which was greater (see student work sample below).



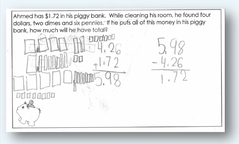
In the work sample, the student represented 1.741 with a large square, seven pieces representing tenths of the square, four pieces representing hundredths and a piece representing a thousandth. The student then represented 1.8 with one large square and eight pieces representing tenths of a square. The student was able to identify 1.8 as the larger quantity by thinking about the combine amount represented by all pieces in each representation.

Instructional cluster 3

- In the final weeks of tutoring, students added and subtracted decimal representations. Students were presented with a decimal placemat and played "Decimal Banker". One of us played the banker and had to give out the tenths, hundredths and thousandths pieces students requested.
- The students were all given a number to represent on the placemats and then were given two additional numbers to add. In order to regroup students had to ask for the specific pieces they would need and had to hand in the correct number of pieces to complete their addition problem.
- Students added numbers with varying place values and even began subtraction with and without borrowing.

For example, in one session each student was given a number (the numbers 3.555, 2.424, and 4.921 were used in this activity) to represent. They then had to add their numbers together on the placemat and regroup. One of the students quickly picked up on the necessity of regrouping and helped the others understand when and why regrouping is needed.

The following session focused on word problems. Students had to add, subtract and compare decimal representations by drawing them. Students initially worked individually and had to determine which operations to use for each problem. Once the problem was completed students shared their work and reasoning with the class. In the work sample below the student visually represented the numbers she found in the problem.



First, she translated the money to a visual decimal representation. Then she added the two numbers using the visual method. Interestingly, she added them vertically and lined up the place values. When asked why, she did not know. She was asked to double-check her work, so she subtracted her final answer from the $4.26 she had translated and found that she was correct.

Post-Assessment Results

Conceptual Understanding Strengths
- Students showed a conceptual understanding of the relationship between fractions and decimals (e.g. 0.1 is equivilent to 1/10).
- Many students showed a strong conceptual understanding of adding decimals together through their understanding of place value and which pieces match up (e.g. 76.21+ 21.34= 97.55).

Conceptual understanding Weaknesses
- Some students were still eager to express that decimals were greater based on the length of the number behind the decimal (e.g. 4.45>4.5 because 45>5).
- One student in particular showed a lack of concept regarding finding the difference between two decimals to the same place value (said 5.5 was only one away from 5.7).

Conceptual understanding is the strand that has shown the greatest amount of progress over the summer. At the beginning of the summer, it was clear that students had very little understanding of decimals, or place value in general. Now, it is clear that they have developed a solid foundation that they will be able to take with them as they enter fifth grade.

Students entered this program fixated on procedures and exhibited very little conceptual knowledge. Now, the procedures that they use are ones that they have developed on their own that are connected to the conceptual understanding they've acquired.

Procedural Fluency Strengths
- All students show a strong ability to represent fractions visually and with numbers
- Students were able to compare decimals with accuracy using the procedures they developed in prior tutoring sessions

Procedural Fluency Weaknesses
- All students at one point or another forgot the "-ths" suffix.
- Students were eager to mix multiple operations and procedures in unsuccessful attempts to solve problems algorithms.

Strategic Competence Strengths
- When presented with a multiplication problem, students could successfully employ the strategy of repeated multiplication(e.g. .47+.47+.47+ .47+ .47+.47+ .47=3.76).
- Students were able to use the strategy of drawing a visual representation of a given decimal in order to solve problems.

Strategic Competence Weaknesses
- One student in particular has the tendency to give up when presented with a problem that he can't do algorithmically.

It is clear that through the development of their conceptual understanding, students have acquired effective strategies to solve the problems they are presented with. We are confident that these problem-solving strategies can be applied to different content areas and will help students become better learners as they progress through school.

At the beginning of tutoring, students were hesitant to verbalize and articulate their thinking. They automatically assumed that if a teacher asked "why," their answer was incorrect. Now, students have gotten into the habit of automatically explaining their thinking without being asked. They are much more confident talking about their answers.

Adaptive Reasoning Strengths
- Students were able to provide a thorough explanation of their understanding of place value in multiple contexts.
- When presented with material that we hadn't covered in previous sessions, students were able to apply their prior knowledge in an attempt to solve the new problem.

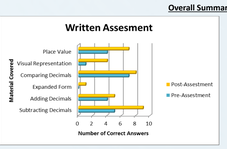
Adaptive Reasoning Weaknesses
- One student in particular was unable to explain her understanding of what the decimal represents in regards to money and place value.
- Some students struggled explaining multiple strategies used to perform arithmetic in regards to decimal representations (e.g. multi-step problems).

Productive Disposition Strengths
- All students could describe the relationship between decimals and money
- Most students expressed confidence in their ability to compare and add decimals.

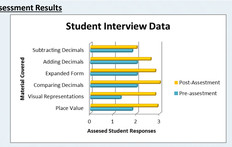
Productive Disposition Weaknesses
- Students have not expressed any additional uses of decimals in the real world.

We hope that in the future, students will be able to gain a more productive disposition in regards to decimals. We tried to incorporate decimals into multiple contexts, but they still are fixated on decimals being used in money in the real-world.

Overall Summary of Post-Assessment Results



The written assessment was 25 multiple choice questions. The data below represents the number of questions answered correctly for each section of material. There were a varying number of questions for each subject matter.



This chart is based on the differences of student response from the first interview to the last interview. The data collected was separated into six sections. Each student was rated on a scale from one to three on their competency in each section. The chart represents the student mean. The scale was separated as such:
1: Student has not worked with the material presented.
2: Student has worked with the material, however, does not know or cannot correctly respond to the interview question.
3: Student has worked with the material and correctly responded to interview question.

Reflection and discussion:

The Common Core State Standards we found most difficult to attain was: CCSS.MATH.CONTENT.5.NBT.B.7

- Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.
- This standard not only includes all four operations, but it includes the connection to a written method as well. That is obviously the natural course of math instruction, but the transfer from the concrete use of manipulatives to the more abstract use of written methods is a very major movement to tackle within the time of our project.
- In this project, we intentionally avoided direct instruction of the algorithm and instead focused on instruction that would foster conceptual understanding. The decimal progression provided by The Common Core State Standards Writing Team (2012) did not stand well on its own. Instead, we used it collectively with other literature (see Theoretical Framework) as well as the learning progressions mentioned in the Fractions section of Common Core State Standards for fourth grade. In order for students to gain a comprehensive understanding of decimal numeration, it is important to pull from these other sources. We found it necessary to draw upon these additional sources because the fourth grade fractions progressions, for example, draw comparisons to the tenths and hundredths grids (see Theoretical Framework), while the fifth grade base ten standards don't use the visual representations we felt necessary.
- Overall, our students benefitted from the use of physical manipulatives, grids and drawing decimal numerations. Whenever they became confused, we could refer them back to their drawings instead of back to an algorithm. Our advice to teachers would be to be patient in two respects. First, we would advise teachers to take the time to develop their own conceptual understanding of the topic. Individuals who have only learned decimal numerations algorithmically will likely become confused when teaching in this conceptual manner. It is important to be able to understand the way students think and know what questions to ask as they progress. Secondly, we would advise teachers that the first part of this progression takes time, but it the end it is a valuable investment. Once students' conceptual understanding is sturdily developed; teaching concepts such as comparing fractions or arithmetic using decimal representations goes much more swiftly. This understanding also allows students to perform tasks more accurately, as well as to develop a higher level of adaptive reasoning. With this foundation, they are much more equipped to explain their thinking and participate in group discussion. Finally, we advise teachers to connect decimals to other mathematical content. It is important for students to see connections between decimals and the real-world (i.e. money, measurement, etc.) as well as to areas such as fractions and percentages.