CONJECTURE $\mathcal{O}$ HOLDS FOR SOME HOROSPHERICAL VARIETIES OF PICARD RANK 1

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Abstract. Property $\mathcal{O}$ for arbitrary complex, Fano manifolds X, is a statement about the eigenvalues of the linear operator obtained from the quantum multiplication of the anticanonical class of X. Pasquier listed the non-homogenous horospherical varieties of Picard rank 1 into five classes. Property $\mathcal{O}$ has already been shown to hold for the odd symplectic Grassmannian which is one class. We will show that Property $\mathcal{O}$ holds for two more classes and an example in a third class of Pasquier's list. The theory of Perron-Frobenius reduces our proofs to be graph theoretic.

1. Introduction

The purpose of this paper is to prove that Conjecture $\mathcal{O}$ holds for some horospherical varieties of Picard rank 1. We recall the precise statement of Conjecture $\mathcal{O}$, following [2, section 3]. Let F be a Fano Variety, let $K := K_F$ be the canonical line bundle of F and let $c_1(F) := c_1(-K) \in H^2(F)$ be the anitcanonical class. The quantum cohomology ring $(QH^*(F), \star)$ is a graded algebra over $\mathbb{Z}[q]$, where q is the quantum parameter. Consider the specialization $H^\bullet(F) := QH^*(F)|_{q=1}$ at $q = 1$. The quantum multiplication by the first Chern class $c_1(F)$ induces an endomorphism $\hat{c}_1$ of the finite-dimensional vector space $H^\bullet(F)$:

$$y \in H^\bullet(F) \mapsto \hat{c}_1(y) := (c_1(F) \star y)|_{q=1}.$$

Denote by $\delta_0 := \max\{|\delta| : \delta \text{ is an eigenvalue of } \hat{c}_1\}$. Then Property $\mathcal{O}$ states the following:

(1) The real number δ_0 is an eigenvalue of $\hat{c}_1$ of multiplicity one.
(2) If δ is any eigenvalue of $\hat{c}_1$ with $|\delta| = \delta_0$, then $\delta = \delta_0\gamma$ for some r-th root of unity $\gamma \in \mathbb{C}$, where r is the Fano index of F.

The property $\mathcal{O}$ was conjectured to hold for any Fano, complex manifold F by Galkin, Golyshev, and Iritani in [2]. If a Fano, complex, manifold has Property $\mathcal{O}$ then we say that the space satisfies Conjecture $\mathcal{O}$.

Next we recall the definition of a horospherical variety following [3]. Let G be a complex reductive group. A G-variety is a reduced scheme of finite type over the field of complex numbers $\mathbb{C}$, equipped with an algebraic action of G. Let B be a Borel subgroup of G. A G-variety X is called spherical if X has a dense B-orbit. Let X be a G-spherical variety and let H be the stabilizer of a point in the dense G-orbit in X. The variety X is called *horospherical* if H contains a conjugate of the maximal unipotent subgroup of G contained in the Borel subgroup B.

Horospherical varieties of Picard rank 1 were classified by Pasquier in [6]. The varieties are either homogeneous or can be constructed in a uniform way via a triple (Type(G),ω_Y,ω_Z)

of representation-theoretic data, where Type(G) is the semisimple Lie type of the reductive group G and ω_Y, ω_Z are the fundamental weights. See [6, Section 1.3] for details. Pasquier classified the possible triples in five classes:

(1) $(B_n, \omega_{n-1}, \omega_n)$ with $n \geq 3$;
(2) $(B_3, \omega_1, \omega_3)$;
(3) $(C_n, \omega_m, \omega_{m-1})$ with $n \geq 2$ and $m \in [2, n]$;
(4) $(F_4, \omega_2, \omega_3)$;
(5) $(G_2, \omega_1, \omega_2)$.

In Propsition 3.6 of [7], Pasquier showed the triples in the above list are Fano varieties. Conjecture $\mathcal{O}$ has already been proved for the homogeneous case by Cheong and Li in [1] and for case (3), the odd symplectic Grassmannian, by Li, Mihalcea, and the last author in [4]. We are now able to state the main theorem:

Theorem 1. If F belongs to the classes (1) for $n = 3$, (2), (3), and (5) of Pasquier's list, then Conjecture $\mathcal{O}$ holds for F.

2. Preliminaries

2.1. Quantum Cohomology. The small quantum cohomology is defined as follows. Let $(\alpha_i)_i$ be a basis of $H^*(F, \mathbb{R})$ and let $(\alpha_i^\vee)_i$ be the dual basis for the Poincaré pairing. The multiplication is given by

$$\alpha_i \star \alpha_j = \sum_{d \geq 0, k} c_{i,j}^{k,d} q^d \alpha_k$$

where $c_{i,j}^{k,d}$ are the 3-point, genus 0, Gromov-Witten invariants corresponding to rational curves of degree d intersecting the classes α_i, α_j, and $\alpha_k^\vee$. We will make use of the quantum Chevalley formula which is the multiplication of a hyperplane class hp with another class a_j. The result [3, Theorem 0.0.3] implies that if F belongs to the classes (1) for $n = 3$, (2), or (5) of Pasquier's list, then there is an explicit quantum Chevalley formula. The explicit quantum Chevalley formula is the key ingredient used to prove Property $\mathcal{O}$ holds.

2.2. Sufficient Criterion for Property $\mathcal{O}$ to hold. We recall the notion of the (oriented) quantum Chevalley Bruhat graph of a Fano variety F. The vertices of this graph are the basis elements $\alpha_i \in H^\bullet(F) := QH^*(F)|_{q=1}$. There is an oriented edge $\alpha_i \to \alpha_j$ if the class α_j appears with positive coeffiecient (we consider $q > 0$) in the quantum Chevalley multiplication $hp \star \alpha_i$ for some hyperplane class hp. The techniques involving Perron-Frobenius theory used by Li, Mihalcea, and Shifler in [4] and Cheong and Li in [1] imply the following lemma:

Lemma 1. If the following conditions hold for a Fano variety F:

(1) the matrix representation of $\hat{c}_1$ is nonnegative,
(2) the quantum Chevalley Bruhat graph of F is strongly connected, and
(3) there exists a cycle of length r, the Fano index, in the quantum Chevalley Bruhat graph of F,

then Property $\mathcal{O}$ holds for F.

We refer the reader to [5, section 4.3] for further details on Perron-Frobenius theory.

3. Checking Property $\mathcal{O}$ Holds

Let X be a horospherical variety. We will simplify our notation where the basis of $H^\bullet(X)$ is $\{1, hp, \alpha_i\}_{i\in I}$ for some finite index set I. Observe by [3] that the anticanonical classes are

$$c_1(X) = \begin{cases} 5hp & \text{when X is case (1) for } n=3 \\ 7hp & \text{when X is case (2)} \\ 4hp & \text{when X is case (5)} \end{cases}$$

and the Fano indices are

$$r = \begin{cases} 5 & \text{when X is case (1) for } n=3 \\ 7 & \text{when X is case (2)} \\ 4 & \text{when X is case (5)} \end{cases}.$$

The endomorphism $\hat{c}_1$ acting on the basis elements of $H^\bullet(X)$ are determined by the Chevalley formula in the following way:

$$\begin{aligned} \hat{c}_1(\alpha_i) &= 5(hp \star \alpha_i)|_{q=1} \text{ when X is case (1) for } n=3, \\ \hat{c}_1(\alpha_i) &= 7(hp \star \alpha_i)|_{q=1} \text{ when X is case (2), and} \\ \hat{c}_1(\alpha_i) &= 4(hp \star \alpha_i)|_{q=1} \text{ when X is case (5).} \end{aligned}$$

Each of the following three subsections will show that Conjecture $\mathcal{O}$ holds for case (1) for $n = 3$, case (2), and case (5) of Pasquier's list, respectively. In each subsection we will reformulate the quantum Chevalley formulas stated in [3], present the quantum Chevalley Bruhat graph, and argue that each condition of Lemma 1 is satisfied. For each case, we have kept the same format of the equations presented by Pech et al. with our prescribed basis for ease of identification for the reader.

3.1. Case (1) for $n = 3$.

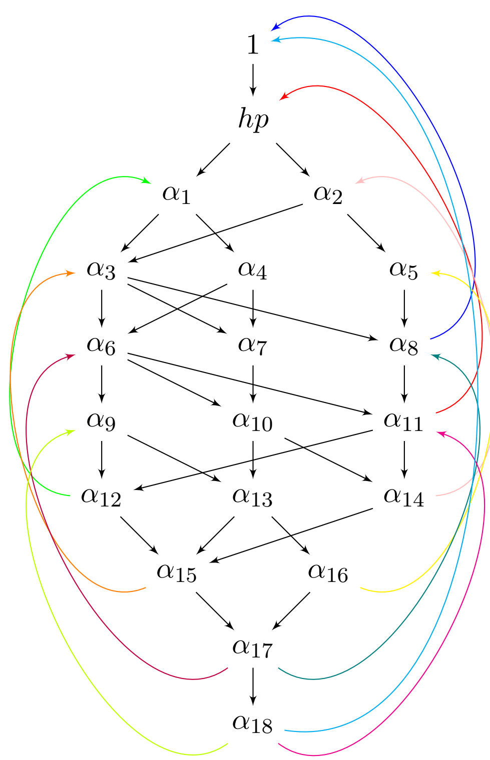
We will reformulate the quantum Chevalley formula stated in [3] using the basis $\{1, hp, \alpha_1, \alpha_2, \cdots, \alpha_{18}\}$.

Proposition 1. *The following equalities hold by* [3, Proposition 4.2.1].

(1) $\hat{c}_1(1) = 5hp$
(2) $\hat{c}_1(hp) = 10\alpha_1 + 5\alpha_2$
(3) $\hat{c}_1(\alpha_1) = 5\alpha_3 + 5\alpha_4$ *and* $\hat{c}_1(\alpha_2) = 10\alpha_3 + 5\alpha_5$
(4) $\hat{c}_1(\alpha_3) = 10\alpha_6 + 5\alpha_7 + 5\alpha_8$, $\hat{c}_1(\alpha_4) = 5\alpha_6 + 10\alpha_7$, *and* $\hat{c}_1(\alpha_5) = 5\alpha_8$
(5) $\hat{c}_1(\alpha_6) = 10\alpha_9 + 5\alpha_{10} + 5\alpha_{11}$, $\hat{c}_1(\alpha_7) = 5\alpha_{10}$ *and* $\hat{c}_1(\alpha_8) = 5\alpha_{11} + 5 \cdot 1$
(6) $\hat{c}_1(\alpha_9) = 5\alpha_{12} + 5\alpha_{13}$, $\hat{c}_1(\alpha_{10}) = 10\alpha_{13} + 5\alpha_{14}$ $\hat{c}_1(\alpha_{11}) = 5\alpha_{12} + 5\alpha_{14} + 5hp$
(7) $\hat{c}_1(\alpha_{12}) = 5\alpha_{15} + 5\alpha_1$, $\hat{c}_1(\alpha_{13}) = 5\alpha_{15} + 5\alpha_{16}$, *and* $\hat{c}_1(\alpha_{14}) = 5\alpha_{15} + 5\alpha_2$
(8) $\hat{c}_1(\alpha_{15}) = 5\alpha_{17} + 5\alpha_3$ *and* $\hat{c}_1(\alpha_{16}) = 5\alpha_{17} + 5\alpha_5$
(9) $\hat{c}_1(\alpha_{17}) = 5\alpha_{18} + 5\alpha_6 + 5\alpha_8$
(10) $\hat{c}_1(\alpha_{18}) = 5\alpha_9 + 5\alpha_{11} + 10 \cdot 1$

The following is the quantum Chevalley Bruhat graph of the Fano variety X in case (1) for $n = 3$. Colored edges are introduced in this figure to improve readability.

FIGURE 1.



Lemma 2. Property $\mathcal{O}$ holds when X is case (1) with $n = 3$ of Pasquier's list.

Proof. The coefficients that appear in the equations in Proposition 1 are the entries of the matrix representation of $\hat{c}_1$. Therefore, the matrix representation of $\hat{c}_1$ is nonnegative. The quantum Chevalley Bruhat graph is strongly connected by Figure 1, and the cycle $\alpha_{18}\alpha_{11}\alpha_{14}\alpha_{15}\alpha_{17}\alpha_{18}$ has length $r = 5$. □

3.2. Case (2). Again, we reformulate the quantum Chavelley formula from [3] using the basis $\{1, hp, \alpha_1, \alpha_2, \cdots, \alpha_{12}\}$.

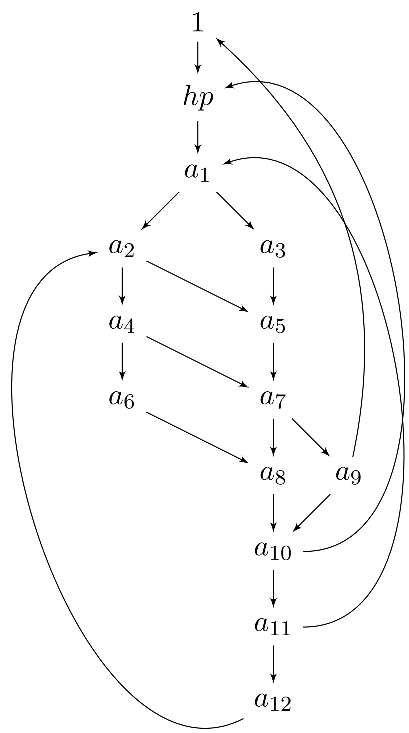
Proposition 2. *The following equalities hold by* [3, Proposition 4.3.1].

(1) $\hat{c}_1(1) = 7hp$
(2) $\hat{c}_1(hp) = 7\alpha_1$
(3) $\hat{c}_1(\alpha_1) = 14\alpha_2 + 7\alpha_3$
(4) $\hat{c}_1(\alpha_2) = 7\alpha_4 + 7\alpha_5$ *and* $\hat{c}_1(\alpha_3) = 7\alpha_5$
(5) $\hat{c}_1(\alpha_4) = 7\alpha_6 + 7\alpha_7$ *and* $\hat{c}_1(\alpha_5) = 7\alpha_7$
(6) $\hat{c}_1(\alpha_6) = 7\alpha_8$ *and* $\hat{c}_1(\alpha_7) = 7\alpha_8 + 7\alpha_9$

(7) $\hat{c}_1(\alpha_8) = 7\alpha_{10}$ *and* $\hat{c}_1(\alpha_9) = 7\alpha_{10} + 7 \cdot 1$
(8) $\hat{c}_1(\alpha_{10}) = 7\alpha_{11} + 7hp$
(9) $\hat{c}_1(\alpha_{11}) = 7\alpha_{12} + 7\alpha_1$
(10) $\hat{c}_1(\alpha_{12}) = 7\alpha_2$

The quantum Chevalley Bruhat graph is

FIGURE 2.



Lemma 3. Property $\mathcal{O}$ holds when X is case (2) of Pasquier's list.

Proof. The coefficients that appear in the equations in Proposition 2 are the entries of the matrix representation of $\hat{c}_1$. Therefore, the matrix representation of $\hat{c}_1$ is nonnegative. The quantum Chevalley Bruhat graph is strongly connected by Figure 2, and the cycle $\alpha_{12}\alpha_2\alpha_4\alpha_6\alpha_8\alpha_{10}\alpha_{11}\alpha_{12}$ has length $r = 7$. □

3.3. Case(5). Again, we reformulate the quantum Chavelley formula from [3] using the basis $\{1, hp, \alpha_1, \alpha_2, \cdots, \alpha_{10}\}$.

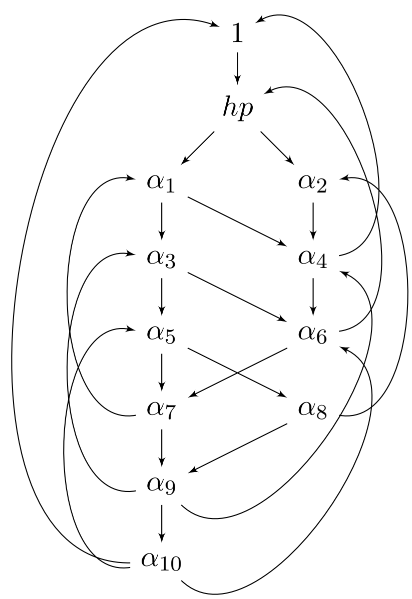
Proposition 3. *The following equalities hold by* [3, Proposition 4.5.1].

(1) $\hat{c}_1(1) = 4hp$
(2) $\hat{c}_1(hp) = 12\alpha_1 + 4\alpha_2$

(3) $\hat{c}_1(\alpha_1) = 8\alpha_3 + 4\alpha_4$ *and* $\hat{c}_1(\alpha_2) = 4\alpha_4$
(4) $\hat{c}_1(\alpha_3) = 12\alpha_5 + 4\alpha_6$ *and* $\hat{c}_1(\alpha_4) = 4\alpha_6 + 4 \cdot 1$
(5) $\hat{c}_1(\alpha_5) = 4\alpha_7 + 4\alpha_8$ *and* $\hat{c}_1(\alpha_6) = 8\alpha_7 + 4hp$
(6) $\hat{c}_1(\alpha_7) = 4\alpha_9 + 4\alpha_1$ *and* $\hat{c}_1(\alpha_8) = 4\alpha_9 + 4\alpha_2$
(7) $\hat{c}_1(\alpha_9) = 4\alpha_{10} + 4\alpha_3 + 4\alpha_4$
(8) $\hat{c}_1(\alpha_{10}) = 4\alpha_5 + 4\alpha_6 + 8 \cdot 1$

The associated quantum Chevalley Bruhat graph is

FIGURE 3.



Lemma 4. Property $\mathcal{O}$ holds when X is case (5) of Pasquier's list.

Proof. The coefficients that appear in the equations in Proposition 3 are the entries of the matrix representation of $\hat{c}_1$. Therefore, the matrix representation of $\hat{c}_1$ is nonnegative. The quantum Chevalley Bruhat graph is strongly connected by Figure 3, and the cycle $\alpha_{10}\alpha_6\alpha_7\alpha_9\alpha_{10}$ has length $r = 4$. □

Theorem 1 follows from Lemmas 2, 3, and 4.

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