CONJECTURE $\mathcal{O}$ HOLDS FOR THE ODD SYMPLECTIC GRASSMANNIAN

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Abstract. Property $\mathcal{O}$, introduced by Galkin, Golyshev and Iritani for arbitrary complex, Fano manifolds X, is a statement about the eigenvalues of the linear operator obtained by the quantum multiplication by the anticanonical class of X. We prove that property $\mathcal{O}$ holds in the case when $X = \mathrm{IG}(k, 2n+1)$ is an odd-symplectic Grassmannian. The proof uses the combinatorics of the recently found quantum Chevalley formula for $\mathrm{IG}(k, 2n+1)$, together with the Perron-Frobenius theory of nonnegative matrices.

1. Introduction

Fix $1 \leqslant k \leqslant n+1$ and let $\mathrm{IG} := \mathrm{IG}(k, 2n+1)$ be the odd-symplectic Grassmannian. This is a smooth Fano algebraic variety parametrizing k dimensional linear subspaces $V \subset \mathbb{C}^{2n+1}$ which are isotropic with respect to a skew-symmetric, bilinear form ω with kernel of dimension 1; see [Mih07, Pecb, MS]. The purpose of this paper is to prove Galkin, Golyshev and Iritani's Conjecture $\mathcal{O}$ [GGI16, Conj. 3.1.2] for the variety IG, i.e. to verify that *Property* $\mathcal{O}$ holds for IG. We recall the precise statement, following [GGI16, §3].

Let $K := K_{\mathrm{IG}}$ be the canonical bundle of IG and let $c_1(\mathrm{IG}) := c_1(-K) \in H^2(\mathrm{IG}) \simeq \mathbb{Z}$ be the anticanonical class. The quantum cohomology ring $(\mathrm{QH}^*(\mathrm{IG}), \star)$ is a graded algebra over $\mathbb{Z}[q]$, where q is the quantum parameter and it has degree $2n+2-k$. Consider the specialization $H^\bullet(\mathrm{IG}) := \mathrm{QH}^*(\mathrm{IG})_{|q=1}$ at $q = 1$. The quantum multiplication by the first Chern class $c_1(\mathrm{IG})$ induces an endomorphism $\hat{c}_1$ of the finite-dimensional vector space $H^\bullet(\mathrm{IG})$:

$$y \in H^\bullet(\mathrm{IG}) \mapsto \hat{c}_1(y) := (c_1(\mathrm{IG}) \star y)|_{q=1}.$$

Denote by $\delta_0 := \max\{|\delta| : \delta \text{ is an eigenvalue of } \hat{c}_1\}$. Then Property $\mathcal{O}$ states the following.

(1) The real number δ_0 is an eigenvalue of $\hat{c}_1$ of multiplicity one.
(2) If δ is any eigenvalue of $\hat{c}_1$ with $|\delta| = \delta_0$, then $\delta = \delta_0 \zeta$ for some r-th root of unity $\zeta \in \mathbb{C}$, where $r = 2n+2-k$ is the Fano index of IG.

Property $\mathcal{O}$ is conjectured to hold for any complex Fano manifold X. It is the main hypothesis needed for the statement of Gamma Conjectures I and II, which in turn are related to mirror symmetry on X and generalize Dubrovin conjectures; we refer to [GGI16] for details. Property $\mathcal{O}$ was proved for several Grassmannians of classical types [Rie01, GG06, Che17] and a complete proof was recently given for any homogeneous space G/P [CL17]. Other known cases include complete intersections in projective spaces [GI, SS, Ke], del Pezzo surfaces [HKLY], few horospherical varieties [BFSS], and a Bott-Samelson threefold [Wit].

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The odd-symplectic Grassmannian IG admits an action of Proctor's (complex) odd-symplectic group Sp_{2n+1} [Pro88]. If $k < n+1$ then Sp_{2n+1} acts with two orbits and if $k = n+1$ the action is transitive and IG is isomorphic to the Lagrangian Grassmannian $\mathrm{IG}(n, 2n)$. The odd-symplectic Grassmannian is sandwiched between two homogeneous spaces

$$\mathrm{IG}(k-1, 2n) \subset \mathrm{IG}(k, 2n+1) \subset \mathrm{IG}(k, 2n+2)$$

where $\mathrm{IG}(k, 2n+2)$ parametrizes the k-dimensional subspaces in $\mathbb{C}^{2n+2}$ which are isotropic with respect to a symplectic form on $\mathbb{C}^{2n+2}$ (and similarly for $\mathrm{IG}(k-1, 2n)$). Then $\mathrm{IG}(k-1, 2n)$ can be identified with the closed orbit under Sp_{2n+1}-action, while $\mathrm{IG}(k, 2n+1)$ is a smooth Schubert variety in $\mathrm{IG}(k, 2n+2)$; see [Mih07, Pecb] and §2 below. An easy exercise is to check this for $k = 1$: then $\mathrm{IG}(1, 2n+1) = \mathbb{P}^{2n}$ and the closed orbit is a single point.

Because quantum cohomology is not functorial, one needs to check Property $\mathcal{O}$ on a case by case basis. In particular, its knowledge for the isotropic Grassmannians $\mathrm{IG}(k-1, 2n)$ and $\mathrm{IG}(k, 2n+2)$ does not imply it for the odd-symplectic Grassmannians IG. Our proof is based on the Perron-Frobenius theory of non-negative matrices, applied to the operator $\hat{c}_1$. The usefulness of this theory for proving Property $\mathcal{O}$ was already noticed in [GGI16, Rmk 3.1.7], and it was the main technique used by Cheong and Li [CL17]. The arguments from [CL17] use that the Gromov-Witten (GW) invariants for G/P are enumerative, in particular the (Schubert) structure constants of $\mathrm{QH}^*(G/P)$ are non-negative integers, and in addition, that the GW invariants satisfy certain symmetries. However, the positivity does not hold for the odd-symplectic Grassmannian (see e.g. (1) below), and it is still unknown whether analogous symmetries exist. We circumvent this problem by making heavy use of the combinatorics of the recently found quantum Chevalley formula in $\mathrm{QH}^*(\mathrm{IG})$ [MS, GPPS, Peca], which governs the quantum multiplication by $c_1(\mathrm{IG})$.

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2. Preliminaries

In this section we briefly introduce the odd-symplectic Grassmanian and some basic properties of its cohomology ring. We refer to [Peca, Pecb] for details; we follow closely the exposition from [MS].

Let $E := \mathbb{C}^{2n+1}$ be an odd dimensional complex vector space with basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{2n+1}\}$. An odd-symplectic form is a skew symmetric, bilinear form ω on E with kernel of dimension 1. W.l.o.g. one can assume that $\ker \omega = \langle \mathbf{e}_1 \rangle$ and that $\omega(\mathbf{e}_i, \mathbf{e}_{2n+3-j}) = \delta_{i,j}$ for $1 \leqslant i \leqslant n+1$ and $2 \leqslant j \leqslant n+1$. The *odd-symplectic Grassmannian* $\mathrm{IG} := \mathrm{IG}(k, 2n+1)$ parametrizes subspaces of dimension k in E which are isotropic with respect to the form ω. It is naturally a subspace of the ordinary Grassmannian $\mathrm{Gr}(k, 2n+1)$, and it is in fact the zero locus of a general section on $\bigwedge^2 \mathcal{S}^*$ induced by the symplectic form ω; here $\mathcal{S}$ denotes the rank k tautological subbundle on $\mathrm{Gr}(k, 2n+1)$. As such it is a projective manifold of dimension

$$\dim \mathrm{IG} = \dim \mathrm{Gr}(k, 2n+1) - \frac{k(k-1)}{2} = k(2n+1-k) - \frac{k(k-1)}{2}.$$

The form ω can be completed to a non-degenerate form $\widetilde{\omega}$ on a space $\mathbb{C}^{2n+2}$, and this gives an embedding $\iota : \mathrm{IG} \to \mathrm{IG}(k, 2n+2)$ into the symplectic Grassmannian which parametrizes linear subspaces isotropic with respect to $\widetilde{\omega}$. The restriction of ω to the subspace $\mathbb{C}^{2n} = \{\mathbf{e}_2, \dots, \mathbf{e}_{2n+1}\}$ is non-degenerate, and this gives an inclusion $\mathrm{IG}(k, 2n) \to \mathrm{IG}(k, 2n+1)$ of the symplectic Grassmannian $\mathrm{IG}(k, 2n)$ into the odd-symplectic one. Therefore one can regard the odd-symplectic Grassmannian as an "intermediate" space between two symplectic

Grassmannians. More is true: the symplectic Grassmannians $\mathrm{IG}(k,2n)$ and $\mathrm{IG}(k,2n+2)$ are homogeneous spaces for the (complex) symplectic groups Sp_{2n} and Sp_{2n+2} respectively. The odd-symplectic Grassmannian has an action of the *odd-symplectic group* Sp_{2n+1}, defined by Proctor [Pro86, Pro88]. This group contains Sp_{2n} as a subgroup, and it is contained in Sp_{2n+2} (but *not* as a subgroup). By definition, the odd-symplectic group is the subgroup of $\mathrm{GL}_{2n+1}(\mathbb{C})$ consisting of those $g \in \mathrm{GL}_{2n+1}(\mathbb{C})$ such that $\omega(g.u, g.v) = \omega(u,v)$ for any $u, v \in \mathbb{C}^{2n+1}$. If $k < n+1$ the group Sp_{2n+1} acts on IG with two orbits given by:

$$X^\circ := \{V \in \mathrm{IG}(k,2n+1) : \mathbf{e}_1 \notin V\}; \quad X_c := \{V \in \mathrm{IG}(k,2n+1) : \mathbf{e}_1 \in V\}.$$

Notice that the closed orbit X_c can be naturally identified with $\mathrm{IG}(k,2n)$. We also remark that if $k=1$ then $\mathrm{IG}(1,2n+1) = \mathbb{P}^{2n}$ (the projective space), while if $k = n+1$ then $\mathrm{IG}(k,2n+1) = \mathrm{IG}(n,2n)$ is the Lagrangian Grassmannian; in the first situation $X_c = \{\langle \mathbf{e}_1 \rangle\}$ (a point), and in the second $X_c = \mathrm{IG}(n,2n)$, thus $X^\circ = \varnothing$. *To avoid special cases, from now on we will consider $k < n+1$.*

Let $P \subset \mathrm{Sp}_{2n+2}$ be the maximal parabolic subgroup which preserves $\mathbf{e}_1$ (i.e.the kernel of ω) and let $B_{2n+2} \subset \mathrm{Sp}_{2n+2}$ be the Borel subgroup which preserves the standard flag in $\mathbb{C}^{2n+2}$. Mihai showed in [Mih07, Prop. 3.3] that there is a surjection $P \to \mathrm{Sp}_{2n+1}$ obtained by restricting $g \mapsto g_{|E}$. Then the Borel subgroup of B_{2n+2} restricts to the (Borel) subgroup $B \subset \mathrm{Sp}_{2n+1}$. We recall the description of B-orbits on IG.

A Schubert variety in $\mathrm{IG}(k,2n+2)$ is the closure of an orbit of the Borel subgroup B_{2n+2}. We follow conventions from [BKT09] and index these Schubert varieties by $(n-k+1)$-strict partitions of the form $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k)$, where $\lambda_1 \leqslant 2n+2-k$ and $\lambda_k \geqslant 0$; the $(n+1-k)$-strict condition means that $\lambda_i > \lambda_{i+1}$ whenever $\lambda_i > n-k+1$. We denote the set of these partitions by Λ^{ev}. For $1 \leqslant i \leqslant n+1$ define $F_i := \langle \mathbf{e}_1, \ldots, \mathbf{e}_i \rangle$ and $F_{n+1+i} = F_{n+1-i}^\perp$, where the perp is taken with respect to the completed (non-degenerate) symplectic form $\widetilde{\omega}$. The Schubert variety $Y(\lambda) \subset \mathrm{IG}(k,2n+2)$ relative to the isotropic flag $F_\bullet = (F_i)$ is defined by

$$Y(\lambda) := \{V \in \mathrm{IG}(k,2n+2) : \dim(V \cap F_{w(j)}) \geqslant j \ \forall 1 \leqslant j \leqslant \ell(\lambda)\}$$

where $\ell(\lambda)$ is the number of non-zero parts of λ and

$$w(j) = 2n+3-k-\lambda_j + \#\{i < j : \lambda_i + \lambda_j : 2(n-k)+j-i\}.$$

This is a subvariety of $\mathrm{IG}(k,2n+2)$ of codimension $|\lambda| := \lambda_1 + \ldots + \lambda_k$. Let 1^k denote the partition $(1,\ldots,1)$. The following key fact is due to Mihai [Mih, Mih07].

Theorem 2.1. *(a) The odd-symplectic Grassmannian $\mathrm{IG}(k,2n+1)$ equals the Schubert variety $Y(1^k)$ in $\mathrm{IG}(k,2n+2)$.*

(b) Those Schubert varieties $Y(\lambda)$ of $\mathrm{IG}(k,2n+2)$ contained in $Y(1^k)$ coincide with the closures of the orbits of the odd-symplectic Borel group B acting on $\mathrm{IG}(k,2n+1)$.

The theorem allows us to define the Schubert varieties in $\mathrm{IG}(k,2n+1)$ as the Schubert varieties in $\mathrm{IG}(k,2n+2)$ contained in $Y(1^k)$. One can check that $Y(\lambda) \subset Y(1^k)$ if and only if λ satisfies the condition that if $\lambda_k = 0$ then $\lambda_1 = 2n+2-k$; in other words, if the first column is not full, then the first row must be full.[1] We will use a variant of the indexing set Λ^{ev}, due to Pech, which conveniently records the codimension relative to IG:

$$\Lambda := \{\lambda = (2n+1-k \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant -1) : \lambda \text{ is } n-k\text{-strict, if } \lambda_k = -1 \text{ then } \lambda_1 = 2n+1-k\}.$$

Pictorially, the partitions in Λ are obtained by removing the full first column 1^k from the partitions in Λ_k^{2n+2}, regardless of whether a part equal to 0 is present. For $\lambda \in \Lambda$, we define

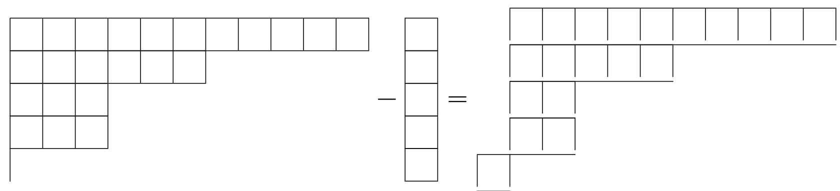
[1] One word of caution: the Bruhat order does not translate into partition inclusion. For example, $(2n+2-k, 0, \ldots, 0) \leqslant (1,1,\ldots 1)$ in the Bruhat order for $k < n+1$.

the Schubert variety $X(\lambda) := Y(\lambda + 1^k)$. Then the codimension of $X(\lambda)$ in $\mathrm{IG}(k, 2n+1)$ equals $|\lambda|$.

Example 2.2. Let $k = 5$, $n = 7$. Consider the partition $(11 \geqslant 6 \geqslant 3 \geqslant 3 \geqslant 0) \in \Lambda^{ev}$. The corresponding partition in Λ is

$$(\lambda_1 - 1 \geqslant \lambda_2 - 1 \geqslant \lambda_3 - 1 \geqslant \lambda_4 - 1 \geqslant \lambda_5 - 1) = (10 \geqslant 5 \geqslant 2 \geqslant 2 \geqslant -1).$$

Pictorially,



2.1. **The (quantum) cohomology ring.** The theorem 2.1 implies that the cohomology ring $H^*(\mathrm{IG})$ of IG has a $\mathbb{Z}$-basis given by the fundamental classes of Schubert varieties $[X(\lambda)] \in H^{2|\lambda|}(\mathrm{IG})$ where λ varies in Λ. The quantum cohomology ring $\mathrm{QH}^*(\mathrm{IG})$ is a graded $\mathbb{Z}[q]$-algebra with a $\mathbb{Z}[q]$-basis given by Schubert classes $[X(\lambda)]$ for $\lambda \in \Lambda$. The grading is given by $\deg q = 2n + 2 - k$ (i.e. the degree of the anticanonical divisor). The multiplication is given by

$$[X(\lambda)] \star [X(\mu)] = \sum_{\nu \in \Lambda, d \geqslant 0} c_{\lambda,\mu}^{\nu,d} q^d [X(\nu)],$$

where $c_{\lambda,\mu}^{\nu,d}$ are the 3-point, genus 0, Gromov-Witten invariants corresponding to rational curves of degree d intersecting the classes $[X(\lambda)]$, $[X(\mu)]$ and the Poincaré dual of $[X(\nu)]$. Unlike the homogeneous case, these numbers might be negative in general. For instance, using Pech's quantum Pieri rule to multiply $[X(\lambda)] \star [X(i)]$ in the case of the odd-symplectic grassmannian of lines $\mathrm{IG}(2, 2n+1)$, one obtains in $\mathrm{QH}^*(\mathrm{IG}(2,5))$,

$$(1)\quad [X(3,-1)] \star [X(2,1)] = -1[X(3,2)] + \ldots; \quad [X(3,-1)] \star [X(3,-1)] = -q[X(0)] + \ldots.$$

For arbitrary k, the second and third named authors found a Chevalley formula calculating $[X(1)] \star [X(\lambda)]$ in the *equivariant* quantum cohomology ring, and proved that this formula gives a recursive algorithm to calculate all the other structure constants; see [MS]. The non-equivariant rule was also found in the recent paper [GPPS].

For the purpose of this paper, we will need the multiplication in the quantum cohomology ring by the Chern class of the anticanonical line bundle $-K := -K_{\mathrm{IG}}$, i.e. by the first Chern class $c_1(\mathrm{IG}(k, 2n+1))$ of the odd-symplectic Grassmannian. A standard calculation yields

$$c_1(\mathrm{IG}) = (2n + 2 - k)[X(1)],$$

therefore the quantum multiplication by $c_1(\mathrm{IG})$ is governed by the Chevalley formula $[X(\lambda)] \star [X(1)]$. Notice also that $X(1)$ is an ample divisor in IG (it is simply the restriction of the Schubert divisor from $\mathrm{Gr}(k, 2n+1)$), therefore $\mathrm{IG}(k, 2n+1)$ is a Fano variety. We refer to either [Peca] or [MS] for details. To describe the quantum multiplication by $[X(1)]$ we need to recall the ordinary Chevalley formula in $H^*(\mathrm{IG}(k, 2n+2))$ proved in [BKT09].

Definition 2.3. *Let $\lambda \in \Lambda^{ev}$. Following* [BKT09, Definitions 1.2, 1.3] *we say that the box in row r and column c of λ is $n+1-k$-related to the box in row r' and column c' if*

$$|c - n + k - 2| + r = |c' - n + k - 2| + r'.$$

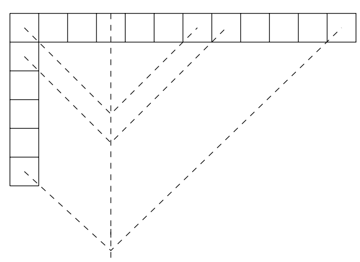
Given $\lambda, \mu \in \Lambda^{ev}$ with $\lambda \subset \mu$, the skew diagram μ/λ is called a horizontal strip (resp. vertical) strip if it does not contain two boxes in the same column (resp. row).

We say that $\lambda \xrightarrow{ev} \mu$ *for any* $n+1-k$*-strict partitions* λ, μ *if* μ *can be obtained by removing a vertical strip from the first* $n+1-k$ *columns of* λ *and adding a horizontal strip to the result, so that*

1. *if one of the first* $n+1-k$ *columns of* μ *has the same number of boxes as the column of* λ*, then the bottom box of this column is* $n+1-k$*-related to at most one box of* $\mu \backslash \lambda$*; and*
2. *if a column of* μ *has fewer boxes than the same column of* λ*, the removed boxes and the bottom box of* μ *in this column must each be* $n+1-k$*-related to exactly one box of* $\mu \backslash \lambda$*, and these boxes of* $\mu \backslash \lambda$ *must all lie in the same row.*

If $\lambda \xrightarrow{ev} \mu$ *, we let* $\mathbb{A}$ *be the set of boxes of* $\mu \backslash \lambda$ *in columns* $n+2-k$ *through* $2n+1-k$ *which are not mentioned in (1) or (2). Then define* $N(\lambda, \mu)$ *to be the number of connected components of* $\mathbb{A}$ *which do not have a box in column* $n+2-k$*. Here two boxes are connected if they share at least a vertex.*

Example 2.4. If μ is obtained from λ by adding exactly one box, then $\lambda \xrightarrow{ev} \mu$. A more interesting example is $\lambda = (2n-2k+2, 1, \ldots, 1)$ (with $k-1$ ones) and $\mu = (2n+2-k)$. In this case each of the boxes in column one of λ is $n+1-k$-related to exactly one box in the first row and last k columns of λ. For instance, consider the case $n = 8$ and $k = 6$. The related boxes are shown in the figure below.



Definition 2.5. *Let* $\lambda, \mu \in \Lambda$ *be two partitions associated to the odd-symplectic Grassmannian* $\mathrm{IG}(k, 2n+1)$*. We say that* $\lambda \to \mu$ *if* $\lambda + 1^k \xrightarrow{ev} \mu + 1^k$ *. If this is the case, we denote by* $A(\lambda, \mu) := N(\lambda + 1^k, \mu + 1^k)$*.*

We need one more definition, for the partitions which will be appear as quantum terms.

Definition 2.6. *Let* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *be a partition in* Λ *such that* $\lambda_1 = 2n+1-k$*.*

(a) If $\lambda_k \geqslant 0$ *then let* $\lambda^* = (\lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_k \geqslant 0)$*. If* $\lambda_k = -1$ *then* λ^* *does not exist.*

(b) If $\lambda_2 = 2n-k$ *then let* $\lambda^{**} = (\lambda_1 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_k \geqslant -1)$*. If* $\lambda_2 < 2n-k$ *then* λ^{**} *does not exist.*

In both situations notice that $|\lambda^*| = |\lambda^{**}| = |\lambda| - (2n+1-k)$. As an example, if $\rho = (2n-k+1, 2n-k, \ldots, 2n-2k+2)$ is the partition indexing the Schubert point, then $\rho^* = (2n-k, \ldots, 2n-2k+2, 0)$ and $\rho^{**} = (2n+1-k, 2n-1-k, \ldots, 2n-2k+2, -1)$. Clearly, there are also examples when one of the partitions λ^* or λ^{**} does not exist, but the other does. We are now ready to state the Chevalley formula proved in [MS] (see also [Pecb, GPPS]).

Theorem 2.7 (quantum Chevalley formula for $\mathrm{IG}(k, 2n+1)$)**.** *Let $\lambda \in \Lambda$ a partition. Then the following holds in* $\mathrm{QH}^*(\mathrm{IG})$*:*

$$[X(1)] \star [X(\lambda)] = \sum_{\lambda \to \mu, |\mu| = |\lambda|+1} 2^{A(\lambda,\mu)}[X(\mu)] + q[X(\lambda^*)] + q[X(\lambda^{**})]$$

If λ^* or λ^{**} do not exist then the corresponding quantum term is omitted.

Consider the operator

$$T = \sum_{i=0}^{\dim \mathrm{IG}} c_1(\mathrm{IG})^{\star i}_{|q=1} = \sum_{i=0}^{\dim \mathrm{IG}} (2n+2-k)^i [X(1)]^{\star i}_{|q=1}, \tag{2}$$

acting on $H^\bullet(\mathrm{IG}) = \mathrm{QH}^*(\mathrm{IG})_{|q=1}$. From the Chevalley formula it follows that if $\lambda \in \Lambda$ is an arbitrary partition then $T[X(\lambda)]$ is an effective combination of Schubert classes. We say that $T[X(\lambda)] > 0$ if in the expansion

$$T[X(\lambda)] = \sum_{\mu \in \Lambda} a(\lambda, \mu)[X(\mu)]$$

all coefficients are strictly positive, i.e. $a(\lambda, \mu) > 0$ for all $\mu \in \Lambda$. Next is a key result in this paper.

Theorem 2.8. *Let $\rho = (2n-k+1, 2n-k, \ldots, 2n-2k+2)$ be the partition indexing the class of the point. Then the following positivity properties hold:*

(a) For any $\lambda \in \Lambda$, the coefficient of $[X(\rho)]$ in $T[X(\lambda)]$ is strictly positive;
(b) The coefficient of $[X(0)]$ in $T[X(\rho)]$ is strictly positive;
(c) $T[X(0)] > 0$.

Before the proof of the theorem we recall the notion of the (oriented) *quantum Bruhat graph* of IG; see [BFP99]. The vertices of this graph consist of partitions $\lambda \in \Lambda$. There is an oriented edge $\lambda \to \mu$ if the class $[X(\mu)]$ appears with positive coefficient (possibly involving q) in the quantum Chevalley multiplication $[X(\lambda)] \star [X(1)]$. An oriented, quantum, *Chevalley chain* between two partitions λ and μ is a chain

$$\lambda_0 := \lambda \to \lambda_1 \to \ldots \to \lambda_s := \mu$$

in the quantum Bruhat graph of $\mathrm{IG}(k, 2n+1)$.

Remark 2.9. Theorem 2.8 implies that the quantum Bruhat graph of IG is strongly connected, i.e. any two of its vertices can be connected by an oriented chain. It is natural to conjecture that $T[X(\lambda)] > 0$ for any λ. If this conjecture is true, it implies that any two points can be connected by a chain containing at most $\dim \mathrm{IG}$ edges.

Proof of Theorem 2.8. In each of parts (a) and (b) it suffices to produce a Chevalley chain between two appropriate partitions which involves at most $\dim \mathrm{IG}$ edges. We consider first the coefficient $a(\lambda, \rho)$. Clearly $a(\lambda, \lambda) \geqslant 1$, therefore we are done if $\lambda = \rho$. If not then one can keep adding exactly one box to produce a Chevalley chain from λ to ρ. If $\lambda_k \geqslant 0$ then it is clear that we arrive at ρ after adding at most $\dim \mathrm{IG}$ boxes. If $\lambda_k = -1$ then necessarily $\lambda_1 = 2n+1-k$, and in the worst case scenario (when $\lambda = (2n+1-k, -1, \ldots, -1)$) we need to add

$$(k-1) + (2n-k) + (2n-k-1) + \ldots + (2n-2k+2) = \dim \mathrm{IG} - (2n-2k+2)$$

boxes. We now turn to part (b). The Chevalley chain from ρ to (0) is constructed as follows. Let η be a partition in Λ of the form $\eta = (\eta_1 \geqslant \ldots \geqslant \eta_k \geqslant 0)$. If $\eta_1 = 2n+1-k$ then $\eta' := \eta^*$ exists and it will be the successor of η. If $\eta_1 < 2n+1-k$ then the successor of η

is $\eta' := (\eta_1 + 1 \geqslant \eta_2 \geqslant \ldots \geqslant \eta_k)$. Notice that in both situations $\eta'_k \geqslant 0$. Now start from $\eta := \rho$ and continue with the rules above. All partitions η in this chain will satisfy $\eta_k \geqslant 0$, and such a chain requires at most

$$k + \sum_{i=1}^{k-1} i = \frac{(k-1)k}{2} = k + \frac{k(k-1)}{2}$$

edges to get to the partition (0). Since

$$\dim \mathrm{IG} = k(2n+1-k) - \frac{(k-1)k}{2} \geqslant \frac{(k-1)k}{2} + k,$$

this completes the proof of (b).

To prove (c) we distinguish two cases, for $\lambda_k \geqslant 0$ and $\lambda_k = -1$. If $\lambda_k \geqslant 0$ then a Chevalley chain from (0) to λ is constructed by successively adding exactly one box, filling rows 1, then 2, etc. Clearly such a chain has exactly $|\lambda| \leqslant \dim \mathrm{IG}$ edges. Assume now that $\lambda_k = -1$. We first construct a chain from (0) to $\alpha := (2n+1-k, -1, \ldots, -1)$ (where there are $k-1$ ones) by

$$(0) \to (1) \to \ldots \to (2n-2k+1) \to \alpha.$$

The last arrow exists by Example 2.4. This chain contains $2n - 2k + 2$ edges. From α to λ one can again add exactly one box at every step, resulting in a Chevalley chain with $|\lambda| - |\alpha| = |\lambda| - (2n - 2k + 2)$ edges. Concatenate the two chains to get a chain from (0) to λ containing $|\lambda|$ edges. □

For future use we also record the following lemma.

Lemma 2.10. *There exists a Chevalley cycle of length $2n + 2 - k$ of the form*

$$(0) \to (1) \to \ldots \to (2n-k+1) \to (0).$$

Proof. This is clear from the Chevalley formula. □

Example 2.11. Consider the case $n = 5$ and $k = 4$. The following illustrates the chain constructed in part (b). (We also include the Chevalley coefficients and quantum parameters.)

$$\begin{array}{ll} (7,6,5,4) & \longrightarrow q(6,5,4,0) \longrightarrow q(7,5,4,0) \longrightarrow q^2(5,4,0,0) \longrightarrow q^2(6,4,0) \longrightarrow 2q^2(7,4,0) \\ \longrightarrow & 2q^2(4,0,0) \longrightarrow 2q^2(5,0,0) \longrightarrow 2^2q^2(6,0,0) \longrightarrow 2^3q^2(7,0,0) \longrightarrow 2^3q^3(0,0,0). \end{array}$$

3. Conjecture $\mathcal{O}$ for $\mathrm{IG}(k, 2n+1)$

In this section we prove the main result of this paper:

Theorem 3.1. *The odd symplectic Grassmannian $\mathrm{IG}(k, 2n+1)$ satisfies Property $\mathcal{O}$, i.e. the quantum multiplication by $c_1(\mathrm{IG}(k, 2n+1))$ satisfies the conditions (1) and (2) stated in §1 above.*

Recall that the Fano index r of IG equals the degree of q, i.e. $r = 2n + 2 - k$. In what follows we fix an arbitrary ordering of the Schubert classes $\{[X_\lambda]\}_\lambda$, and let M denote the matrix of $\hat{c}_1$ with respect to such an ordered basis. The quantum Chevalley rule implies that M is a *nonnegative* matrix, i.e. all its coefficients are nonnegative. The theory of nonnegative matrices (see e.g. in [Min88]) will play a fundamental role. We refer to [CL17, §3.1 and §3.2] for more details and the context of the facts needed in what follows.

Lemma 3.2. *The matrix M is irreducible in the sense that PMP^t is never of the form* $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ *for any permutation matrix P, where A, D are square submatrices.*

Proof. If there exists a permutation matrix P such that PMP^t is a block-upper triangular matrix, then so is $\sum_{m=0}^{\dim X} PM^mP^t$. The matrix of the operator T is nonnegative, and since M is reducible it follows that (the matrix of) T is again reducible. By a remarkable property of reducible nonnegative matrices (see e.g. [CL17, Remark 3.1, part (1)]) T must preserve a proper *coordinate* subspace V. Let $[X(\lambda)] \in V$ be a Schubert class inside this subspace. Another remarkable property is that if a class $[X(\mu)]$ appears with positive coefficient in the expansion of $T[X(\lambda)]$ then $[X(\mu)] \in V$ (again see [CL17, remark 3.1, part (2)]). But then by part (a) of Theorem 2.8 it follows that $[X(\rho)] \in V$, part (b) implies that $[X(0)] \in V$, and part (c) implies that any other $[X(\mu)] \in V$. In particular, $V = H^\bullet(\mathrm{IG})$, which is a contradiction. □

According to Perron-Frobenius theory, any irreducible matrix has a real, positive eigenvalue δ_0 of multiplicity one such that for any other eigenvalue δ there is an inequality $\delta_0 \geqslant |\delta|$; cf. [BP94, Thm. 1.4]. In order to show that M satisfies Property $\mathcal{O}$ we need to study the *index of imprimitivity* of M. We recall next the relevant definitions, following [CL17].

Definition 3.3. *(a) Let A be an irreducible $n \times n$ matrix with maximal eigenvalue δ_0, and suppose that A has exactly h eigenvalues of modulus $|\delta_0|$. The number h is called the index of imprimitivity of A.*

(b) Two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ are said to have the same zero pattern if $a_{i,j} = 0$ if and only if $b_{i,j} = 0$. A directed graph $D(A)$ is said to be associated with a nonnegative matrix A, if the adjacency matrix of $D(A)$ has the same zero pattern as A.

(c) Let D be a strongly connected directed graph. The greatest common divisor of the lengths of all cycles in D is called the index of imprimitivity of D .

To any pair of a nonnegative matrix A and an *ordered* basis one can define a directed graph $D(A)$ such that $D(A)$ is associated to A. This is done by replacing nonzero entries of A by 1, and considering the resulting matrix as the adjacency matrix of a directed graph; the direction of the arrows are determined by the ordering of the basis; see e.g. [CL17, §3.2]. In our situation the graph $D(M)$ is simply the oriented, quantum Bruhat graph defined in the previous section. Next is a key result relating the index of imprimitivity of M to that of an associated directed graph; see Theorems 3.2 and 3.3 of Chapter 4 of [Min88].

Proposition 3.4. *Let A be a nonnegative matrix and $D(A)$ the associated directed graph defined above. Then the following hold:*

(a) A is irreducible if and only if the associated directed graph $D(A)$ is strongly connected;

(b) If A is irreducible, then the index of imprimitivity $h(A)$ of A is equal to the index of imprimitivity $h(D(A))$ of the associated directed graph $D(A)$.

Proof of Theorem 3.1. Let h denote the imprimitivity of matrix M. The eigenvalues of $\hat{c}_1$ are that of the matrix M. Since M is an irreducible, nonnegative matrix, the results [BP94, Theorems 1.4 and 2.20 of Chapter 2] imply the following two facts.

(i) The real number $\delta_0 := \max\{|\delta| : \delta \text{ is an eigenvalue of } \hat{c}_1\}$ is an eigenvalue of $\hat{c}_1$ of multiplicity one.

(ii) Denote by $\delta_1, \cdots, \delta_h$ all eigenvalues of M of modulus δ_0 with multiplicities counted. Then $\{\frac{\delta_i}{\delta_0} \mid 1 \leqslant i \leqslant h\}$ are precisely the h-th roots of unity.

Part (i) proves condition (1) of the Property $\mathcal{O}$. To prove the second condition of Property $\mathcal{O}$ it suffices to show that $h = r$ (the Fano index). A general property of Fano manifolds shows that r always divides h; see [GGI16, Remark 3.1.3]. For the converse, notice that by

Lemma 2.10 there exists a cycle of length $r = 2n + 2 - k$ in $D(M)$. Then h divides r by Proposition 3.4, hence $h = r$ and we are done. □

Remark 3.5. The key ingredients that prove Conjecture $\mathcal{O}$ holds for any homogeneous space G/P [CL17] and for the odd symplectic Grassmannian include the three observations:

(1) The matrix representation of the endomorphism $\hat{c}_1$ is nonnegative;
(2) the quantum Bruhat graph is strongly connected;
(3) the quantum Bruhat graph has a set of cycles such that the greatest common divisor of their lengths equals to to r, the Fano index.

These observations were used in the recent paper [BFSS] and may be useful to prove that Conjecture $\mathcal{O}$ holds for other Fano varieties. Notice that the first listed item is not necessary for Conjecture $\mathcal{O}$ to hold (see e.g. [Wit] and [HKLY]).

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