

# ON THE SPECTRAL PROPERTIES OF THE QUANTUM COHOMOLOGY OF ODD QUADRICS

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ABSTRACT. Let  $H^\bullet(\text{OG})$  be the quantum cohomology (specialized at  $q = 1$ ) of the  $2n - 1$  dimensional quadric  $\text{OG}$ . We will calculate the characteristic polynomial of the linear operators induced by quantum multiplication in  $H^\bullet(\text{OG})$  and the Frobenius-Perron dimension. We also check that Galkin's lower bound conjecture holds for  $\text{OG}$ .

## 1. INTRODUCTION

Let  $\text{OG} := \text{OG}(1, 2n + 1)$  be the  $2n - 1$  dimensional quadric. This is the parameterization of isotropic 1 dimensional subspaces of  $\mathbb{C}^{2n+1}$ , isotropic with respect to a symmetric non-degenerate symmetric form. The quantum cohomology ring  $(\text{QH}^*(\text{OG}), \star)$  is a graded algebra over  $\mathbb{Z}[q]$ , where  $q$  is the quantum parameter and  $\deg q = 2n - 1$ . The ring  $(\text{QH}^*(\text{OG}), \star)$  has Schubert classes given by  $\tau_p$ , with  $0 \leq p \leq 2n - 1$ . The ring multiplication is given by

$$\tau_p \star \tau_i = \sum_{j, d_0 \geq 0} c_{p,i}^{j, d_0} q^{d_0} \tau_j$$

where  $c_{p,i}^{j, d_0}$  is the Gromov-Witten invariant that enumerates degree  $d_0$  rational curves intersecting general translate of  $\tau_p, \tau_i$ , and the Poincaré dual of  $\tau_j$  which is  $\tau_{2n-1-j}$ .

Consider the specialization  $H^\bullet(\text{OG}) := \text{QH}^*(\text{OG})|_{q=1}$  at  $q = 1$ . The quantum multiplication by the Schubert class  $\tau_p$ , with  $0 \leq p \leq 2n - 1$ , induces an endomorphism  $A(\tau_p)$  of the finite-dimensional vector space  $H^\bullet(\text{OG})$ :

$$y \in H^\bullet(\text{OG}) \mapsto A(\tau_p)(y) := (\tau_p \star y)|_{q=1}.$$

Denote by  $\delta_p := \max\{|\delta| : \delta \text{ is an eigenvalue of } A(\tau_p)\}$  the real largest eigenvalue of  $A(\tau_p)$ . Define the Frobenius-Perron Dimension to be

$$\text{FPdim} : H^\bullet(\text{OG}) \rightarrow \mathbb{C} \text{ defined by } \text{FPdim}(\tau_p) = \delta_p.$$

The first study of Frobenius-Perron dimension on quantum cohomology specialized at  $q = 1$  is for the Grassmannian in [Rie01] with subsequent study in [LSYZ].

To simplify notation we let  $d := \text{GCD}(p, 2n - 1)$  for the remainder of the article. We will prove the next two results.

**Theorem 1.1.** *The characteristic polynomial of  $A(\tau_p)$  is*

$$\begin{cases} \lambda \left( \lambda^{\frac{2n-1}{d}} - 2^{\frac{2p}{d}} \right)^d & 1 \leq p < n \\ \lambda \left( \lambda^{\frac{2n-1}{d}} - 2^{\frac{2p-(2n-1)}{d}} \right)^d & n \leq p < 2n - 1 \\ (\lambda - 1)^{2n-1} (\lambda + 1) & p = 2n - 1 \end{cases}$$

*In particular, the eigenvalues of  $A(\tau_p)$  are simple if and only if the integers  $p$  and  $2n - 1$  are relatively prime.*

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2010 *Mathematics Subject Classification.* Primary 14N35; Secondary 15B48, 14N15, 14M15.

The next corollary follows immediately from Theorem 1.1.

**Corollary 1.2.** *We have that  $FPdim(\tau_p) = \begin{cases} 2^{\frac{2p}{2n-1}} & 1 \leq p < n \\ 2^{\frac{2p}{2n-1}-1} & n \leq p < 2n-1 \\ 1 & p = 2n-1. \end{cases}$*

The final result will be a proof of a lower bound on the real largest eigenvalue of the first Chern class. We will state this precisely next. Let  $K := K_{OG}$  be the canonical bundle of OG and let  $c_1 := c_1(-K) \in H^2(OG)$  be the anticanonical class. Galkin's lower bound conjecture for the OG case<sup>1</sup> states that (see [Gal]):

$$FPdim(A(c_1)) \geq \dim_{\mathbb{C}} OG + 1.$$

Here,  $\dim_{\mathbb{C}} OG = 2n - 1$  and  $c_1 = (2n - 1)\tau_1$ .

**Theorem 1.3.** *Galkin's lower bound conjecture holds for OG.*

The result is proved for other varieties in [ESS+20, LSYZ, CH20, Ke].

*Acknowledgements.* Shifler is partially supported by the Building Research Excellence (BRE) Program at Salisbury University. The authors would like to thank the reviewer for carefully reading the manuscript and providing useful comments and recommendations.

## 2. PRELIMINARIES

We will rigorously define OG. The *reverse dot product* of two vectors  $(a_1, a_2 \dots a_n)$  and  $(b_1, b_2, \dots b_n)$  is

$$(1) \quad ((a_1, a_2 \dots a_n), (b_1, b_2, \dots b_n)) = \sum_{i=1}^n a_i b_{n+1-i}.$$

We are ready to define OG.

**Definition 2.1.** *The odd dimensional quadric OG is*

$$OG = \{V \subset \mathbb{C}^{2n+1} : \dim V = 1, (a, b) = 0 \forall a, b \in V\}.$$

*Equivalently, OG is the loci of  $v_{n+1}^2 + 2 \sum_{i=1}^n v_i v_{2n+2-i}$ .*

**2.1. Schubert cells and varieties.** The Schubert cells and Schubert varieties are denoted by  $X(p)^\circ$  and  $X(p)$ , respectively, with  $0 \leq p \leq 2n - 1$ . We will first state what the Schubert cells are and follow with a description of the Schubert varieties.

The Schubert cells are given as follows.

(1) For  $0 \leq p \leq n$ , we have that

$$X(p)^\circ = \left\{ \text{span}\left\{ \overbrace{(0, \dots, 0)}^p, 1, \overbrace{(*, \dots, *)}^{2n-1-2p}, \bullet, \overbrace{(*, \dots, *)}^p \right\} : * \text{ is free and } \bullet \text{ is given by (1)} \right\}.$$

(2) For  $n + 1 \leq p \leq 2n - 1$ , we have that

$$X(p)^\circ = \left\{ \text{span}\left\{ \overbrace{(0, \dots, 0)}^{p+1}, 1, \overbrace{(*, \dots, *)}^{2n-1-p} \right\} : * \text{ is free} \right\}.$$

<sup>1</sup>Galkin's lower bound conjecture is for any fano variety.

The Schubert variety  $X(p)$  is the closure of the Schubert cell  $X(p)^\circ$ . Precisely, the Schubert variety  $X(p)$  is given by

$$X(p) = \bigcup_{i \geq p} X(i)^\circ.$$

Furthermore,  $\dim X(p) = \dim X(p)^\circ = (2n - 1) - p$ .

**2.2. Quantum cohomology.** We will now introduce the quantum cohomology ring of OG. The (small) quantum cohomology of OG, denoted by  $\text{QH}^*(\text{OG})$ , has a graded  $\mathbb{Z}[q]$ -basis consisting of Schubert classes  $\tau_p := [X(p)]$  where  $0 \leq p \leq 2n - 1$  and  $\deg q = 2n - 1$ . The ring multiplication is given by

$$\tau_p \star \tau_i = \sum_{j, d_0 \geq 0} c_{p,i}^{j, d_0} q^{d_0} \tau_j$$

where  $c_{p,i}^{j, d_0}$  is the Gromov-Witten invariant that enumerates degree  $d_0$  rational curves intersecting general translate of  $\tau_p, \tau_i$ , and the Poincaré dual of  $\tau_j$  which is  $\tau_{2n-1-j}$ . We refer the reader to [FP97, KM94] for additional details regarding quantum cohomology.

The quantum Chevalley formula for OG is given as follows (see [BKT, BKT09]).

$$\tau_1 \star \tau_p = \begin{cases} \tau_{p+1} & 1 \leq p \leq n - 1 \\ 2\tau_{n+1} & p = n \\ \tau_{p+1} & n + 1 \leq p \leq 2n - 3 \\ \tau_{2n-1} + q\tau_0 & p = 2n - 2 \\ q\tau_1 & p = 2n - 1. \end{cases}$$

Consider the specialization  $H^\bullet(\text{OG}) := \text{QH}^*(\text{OG})|_{q=1}$  at  $q = 1$ . The quantum multiplication by  $\tau_p$  induces an endomorphism  $A(\tau_p)$  of the finite-dimensional vector space  $H^\bullet(\text{OG})$ :

$$y \in H^\bullet(\text{OG}) \mapsto A(\tau_p)(y) := (A(\tau_p) \star y)|_{q=1}.$$

*Example 2.2.* The matrix  $A(\tau_1)$  for  $\text{OG}(1, 3)$  is computed below.

$$A(\tau_1) = \begin{matrix} & \tau_0 & \tau_1 & \tau_2 & \tau_3 \\ \tau_0 & \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \\ \tau_1 & \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \\ \tau_2 & \begin{pmatrix} 0 & 2 & 0 & 0 \end{pmatrix} \\ \tau_3 & \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

The next proposition follows from immediately from the quantum Chevalley formula.

**Proposition 2.3.** *For  $0 \leq p \leq 2n - 1$ , we have that*

$$A(\tau_p) = \begin{cases} A(\tau_1)^p & 1 \leq p \leq n - 1 \\ \frac{1}{2}A(\tau_1)^p & n \leq p \leq 2n - 2 \\ \frac{1}{2}A(\tau_1)^{2n-1} - A(\tau_0) & p = 2n - 1. \end{cases}$$

*Recall that  $A(\tau_0)$  is the identity matrix.*

3. SPECTRAL PROPERTIES OF  $A(\tau_1)$ 

We will study the Spectral properties of  $A(\tau_1)$  in this section.

**Proposition 3.1.** *The characteristic polynomial for the linear operator  $A(\tau_1)$  induced from the quantum multiplication in  $\text{QH}^*(\text{OG})$  is  $\lambda^{2n} - 4\lambda$ .*

*Proof.* The basis of this proof is to swap the rows around, moving the top row to the bottom and shifting the rest up by one. This will allow us to get an “almost” diagonal matrix of the form:

$$\begin{matrix} r_1 \\ r_{2+i} \\ r_n \\ r_{n+1} \\ r_{n+2} \\ r_{n+2+j} \\ r_{2n-1} \\ r_{2n} \end{matrix} \begin{pmatrix} 1 & -\lambda & & & & & & 1 \\ & \ddots & \ddots & & & & & \\ & & 1 & -\lambda & & & & \\ & & & 2 & -\lambda & & & \\ & & & & 1 & -\lambda & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & -\lambda \\ -\lambda & & & & & & 1 & 0 \end{pmatrix}$$

We can reduce this matrix to a diagonal matrix by subtracting a factor times each row from the last row. We will use the property that if two rows (or two columns) of a matrix are interchanged, the value of the determinant changes sign. With the appropriate row operations we can get a matrix of the form:

$$\begin{matrix} r_1 \\ r_{2+i} \\ r_n \\ r_{n+1} \\ r_{n+2} \\ r_{n+2+j} \\ r_{2n-1} \\ r_{2n} \end{matrix} \begin{pmatrix} 1 & -\lambda & & & & & & 1 \\ & \ddots & \ddots & & & & & \\ & & 1 & -\lambda & & & & \\ & & & 2 & -\lambda & & & \\ & & & & 1 & -\lambda & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & -\lambda \\ & & & & & & & \frac{4\lambda - \lambda^{2n}}{2} \end{pmatrix}$$

This is an upper triangular matrix so its determinant is the product of its diagonal given by  $1 \cdot 1 \cdot \dots \cdot 1 \cdot 2 \cdot 1 \cdot \dots \cdot 1 \cdot \frac{4\lambda - \lambda^{2n}}{2} = 4\lambda - \lambda^{2n}$ . Finally since we swapped around an odd number of rows ( $2n - 1$  swaps were made) we need to multiply this determinant by a factor of  $-1$  to get the determinant of  $\tau_1 - \lambda I$  to be  $\lambda^{2n} - 4\lambda$ . Therefore the characteristic polynomial of  $\tau_1$  is  $\lambda^{2n} - 4\lambda$ .  $\square$

Recall that the eigenvalues of a matrix are the roots of its characteristic polynomial so we can factor the characteristic polynomial to get eigenvalues of the form

$$\lambda \in \left\{ \sqrt[2n-1]{4e^{\frac{2\pi j i}{2n-1}}} : 0 \leq j \leq 2n - 1 \right\} \cup \{0\}.$$

The next Corollary follows immediately from Proposition 3.1.

**Corollary 3.2.** *If  $\lambda$  is an eigenvalue of the linear operator  $\tau_1$  induced from the quantum multiplication in  $\text{QH}^*(\text{OG})$  then  $\frac{\lambda^{2n-1}-2}{2} = \begin{cases} -1 & \lambda = 0 \\ 1 & \text{otherwise} \end{cases}$ .*

We can use this corollary to prove the following theorem.

**Theorem 3.3.** *The eigenvector of  $A(\tau_1)$  associated to the eigenvalue  $\lambda_j = \sqrt[2n-1]{4e^{\frac{2\pi j i}{2n-1}}}$ , for some  $1 \leq j \leq 2n-1$ , or  $\lambda_0 = 0$  is given as follows with respect to the basis  $\{\tau_i\}_{0 \leq i \leq 2n-1}$ .*

$$\vec{v}_j = \left( \frac{\lambda_j^{2n-1} - 2}{2}, \frac{\lambda_j^{2n-2}}{2}, \frac{\lambda_j^{2n-3}}{2}, \dots, \frac{\lambda_j^n}{2}, \lambda_j^{n-1}, \lambda_j^{n-2}, \dots, \lambda_j, 1 \right).$$

Finally, we have that  $A(\tau_1) = PDP^{-1}$  where  $P = [\vec{v}_0^t \ \vec{v}_1^t \ \dots \ \vec{v}_{2n-1}^t]$  and  $D = \text{diag}(\lambda_0, \lambda_1, \dots, \lambda_{2n-1})$ .

*Proof.* First observe that we have the following equality where the vectors on the right are written with respect to the basis  $\{\tau_i\}_{0 \leq i \leq 2n-1}$ .

$$\begin{aligned} \tau_0 + \tau_{2n-1} + \frac{1}{2} \sum_{k=1}^{n-1} \lambda_j^k \tau_{2n-1-k} + \sum_{k=n}^{2n-2} \lambda_j^k \tau_{2n-1-k} &= \left( 1, \frac{\lambda_j^{2n-2}}{2}, \frac{\lambda_j^{2n-3}}{2}, \dots, \frac{\lambda_j^n}{2}, \lambda_j^{n-1}, \lambda_j^{n-2}, \dots, \lambda_j, 1 \right); \\ \tau_{2n-1} - \tau_0 &= (-1, 0, 0, \dots, 0, 0, 1). \end{aligned}$$

Using Corollary 3.2 we can write

$$\vec{v}_j = \left( \frac{\lambda_j^{2n-1} - 2}{2}, \frac{\lambda_j^{2n-2}}{2}, \frac{\lambda_j^{2n-3}}{2}, \dots, \frac{\lambda_j^n}{2}, \lambda_j^{n-1}, \lambda_j^{n-2}, \dots, \lambda_j, 1 \right).$$

To prove that our vector  $\vec{v}_j^t$  is an eigenvector of  $A(\tau_1)$  with corresponding eigenvalue  $\lambda_j$  we need to multiply  $A(\tau_1)$  by  $\vec{v}_j^t$ . To multiply the matrix by the vector we can take the dot product of each row of  $A(\tau_1)$  with the vector  $\vec{v}_j^t$ . We index the rows of  $A(\tau_1)$  by  $r_i$  using the same conventions as those in Proposition 3.1.

A direct calculation show that  $r_1 \cdot \vec{v}_j^t = \lambda_j \frac{\lambda_j^{2n-1} - 2}{2}$  and  $r_i \cdot \vec{v}_j^t = \lambda_j \cdot \frac{\lambda_j^{2n-i}}{2}$  for  $2 \leq i \leq 2n$ . Thus,  $A(\tau_1)\vec{v}_j^t = \lambda_j \vec{v}_j^t$ . The result follows.  $\square$

#### 4. SPECTRAL PROPERTIES OF $A(\tau_p)$

We will study the Spectral properties of  $A(\tau_p)$  in this section. Recall that

$$A(\tau_p) = \begin{cases} A(\tau_1)^p & 1 \leq p \leq n-1 \\ \frac{1}{2}A(\tau_1)^p & n \leq p \leq 2n-2 \\ \frac{1}{2}A(\tau_1)^{2n-1} - A(\tau_0) & p = 2n-1. \end{cases}$$

**Lemma 4.1.** *We have the following results.*

- (1) *The set of matrices  $A(\tau_p)$ , with  $0 \leq p \leq 2n-1$ , are simultaneously diagonalizable.*
- (2) *For  $1 \leq p \leq n-1$ , the eigenvalues of  $A(\tau_p)$  are  $0, \lambda_1^p, \lambda_2^p, \dots, \lambda_{\frac{2n-1}{d}}^p$  where each nonzero eigenvalue has multiplicity  $d$ .*
- (3) *For  $n \leq p \leq 2n-2$ , the eigenvalues of  $A(\tau_p)$  are  $0, \frac{1}{2}\lambda_1^p, \frac{1}{2}\lambda_2^p, \dots, \frac{1}{2}\lambda_{\frac{2n-1}{d}}^p$  where each nonzero eigenvalue has multiplicity  $d$ .*
- (4) *The eigenvalues of  $A(\tau_{2n-1})$  are  $1$ , with multiplicity  $2n-1$ , and  $-1$ , with multiplicity  $1$ .*

*Proof.* First observe that that  $A(\tau_0) = I$  is the identity matrix. The result follows from the following obvious facts.

- (1)  $A(\tau_p) = A(\tau_1)^p = PD^pP^{-1}$  for  $1 \leq p \leq n-1$ ;
- (2)  $A(\tau_p) = \frac{1}{2}A(\tau_1)^p = \frac{1}{2}PD^pP^{-1}$  for  $n \leq p \leq 2n-2$ ;

(3) and for  $p = 2n - 1$  we have that

$$A(\tau_{2n-1}) = \frac{1}{2}PD^{2n-1}P^{-1} - PIP^{-1} = P\left(\frac{1}{2}D^{2n-1} - I\right)P^{-1}.$$

□

We will next present the proof of Theorem 1.1.

*Proof.* Clearly by Lemma 4.1, we know that the characteristic polynomial of  $A(\tau_{2n-1})$  is  $(\lambda - 1)^{2n-1}(\lambda + 1)$ .

By Lemma 4.1, we have that the eigenvalues of  $A(\tau_p)$ , for  $p < n$ , are  $4^{\frac{p}{2n-1}} \cdot \left(e^{2\pi i \frac{p}{2n-1}}\right)^j$  (for  $0 \leq j \leq 2n - 1$ ) with each having multiplicity  $\text{GCD}(p, 2n - 1)$ . Similarly, for  $n \leq p < 2n - 1$ , the eigenvalues of  $A(\tau_p)$  are  $4^{\frac{p}{2n-1}-1} \cdot \left(e^{2\pi i \frac{p}{2n-1}}\right)^j$  (for  $0 \leq j \leq 2n - 1$ ) with each having multiplicity  $\text{GCD}(p, 2n - 1)$ .

The characteristic polynomial of  $A(\tau_p)$ , for  $p < n$ , is

$$\begin{aligned} \lambda \prod_{k=0}^{d-1} \prod_{j=0}^{\frac{2n-1}{d}-1} \left( \lambda - 4^{\frac{p}{2n-1}} \left( e^{\frac{2\pi i j}{2n-1}} \right) \right) &= \lambda \prod_{k=0}^{d-1} \left( \lambda^{\frac{2n-1}{d}} - 4^{\frac{p}{2n-1} \cdot \frac{2n-1}{d}} \right) \\ &= \lambda \left( \lambda^{\frac{2n-1}{d}} - 4^{\frac{p}{d}} \right)^d \\ &= \lambda \left( \lambda^{\frac{2n-1}{d}} - 2^{\frac{2p}{d}} \right)^d \end{aligned}$$

The characteristic polynomial of  $A(\tau_p)$ , for  $n \leq p < 2n - 1$ , is

$$\begin{aligned} &\lambda \prod_{k=0}^{d-1} \prod_{j=0}^{\frac{2n-1}{d}-1} \left( \lambda - \frac{1}{2} \cdot 4^{\frac{p}{2n-1}} \left( e^{\frac{2\pi i j}{2n-1}} \right) \right) \\ &= \lambda \prod_{k=0}^{d-1} \left( \lambda^{\frac{2n-1}{d}} - \left( \frac{1}{2} \cdot 4^{\frac{p}{2n-1}} \right)^{\frac{2n-1}{d}} \right) \\ &= \lambda \prod_{k=0}^{d-1} \left( \lambda^{\frac{2n-1}{d}} - \left( \frac{1}{2} \right)^{\frac{2n-1}{d}} \cdot \left( 4^{\frac{p}{d}} \right) \right) \\ &= \lambda \prod_{k=0}^{d-1} \left( \lambda^{\frac{2n-1}{d}} - \left( 4^{-\frac{2n-1}{2d}} \right) \cdot \left( 4^{\frac{p}{d}} \right) \right) \\ &= \lambda \prod_{k=0}^{d-1} \left( \lambda^{\frac{2n-1}{d}} - \left( 2^{-\frac{2n-1}{d}} \right) \cdot \left( 2^{\frac{2p}{d}} \right) \right) \\ &= \lambda \prod_{k=0}^{d-1} \left( \lambda^{\frac{2n-1}{d}} - \left( 2^{\frac{2p-(2n-1)}{d}} \right) \right) \\ &= \lambda \left( \lambda^{\frac{2n-1}{d}} - 2^{\frac{2p-(2n-1)}{d}} \right)^d \end{aligned}$$

The result follows. □

## 5. GALKIN'S LOWER BOUND CONJECTURE FOR OG

Lastly we check that Galkin's Lower Bound Conjecture holds for  $H^\bullet(\text{OG})$ .

**Theorem 5.1.** *Galkin's Lower Bound Conjecture holds for  $H^\bullet(\text{OG})$ . That is,*

$$FPdim((2n-1)\tau_1) \geq \dim \text{OG} + 1.$$

*Proof.* The smooth real function  $f(x) := 4^x - x - 1$  satisfies  $f(0) = 0$  and  $f'(x) = 4^x \ln(4) - 1 > 0$ . Hence,  $f\left(\frac{1}{2n-1}\right) > 0$ . That is,  $4^{\frac{1}{2n-1}} > \frac{1}{2n-1} + 1 = \frac{2n}{2n-1}$ . The result follows.  $\square$

## REFERENCES

- [BKT] Anders Buch, Andrew Kresch, and Harry Tamvakis. Quantum Giambelli formulas for isotropic Grassmannians. *Math. Ann.*, 354(3):801–812.
- [BKT09] Anders Buch, Andrew Kresch, and Harry Tamvakis. Quantum Pieri rules for isotropic Grassmannians. *Invent. Math.*, 178(2):345–405, 2009.
- [CH20] Daewoong Cheong and Manwook Han. Galkin's lower bound conjecture for lagrangian and orthogonal grassmannians. *Bulletin of the Korean Mathematical Society*, 57(4):933–943, 2020.
- [ESS<sup>+</sup>20] La'Tier Evans, Lisa Schneider, Ryan M. Shifler, Laura Short, and Stephanie Warman. Galkin's lower bound conjecture holds for the grassmannian. *Communications in Algebra*, 48(2):857–865, 2020, <https://doi.org/10.1080/00927872.2019.1663383>.
- [FP97] William Fulton and Rahul Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.
- [Gal] Sergey Galkin. The conifold point. arXiv:1404.7388.
- [Ke] Hua-Zhong Ke. On Conjecture  $\mathcal{O}$  for projective complete intersections. available on arXiv:1809.10869.
- [KM94] Maxim Kontsevich and Yuri Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. *Comm. Math. Phys.*, 164(3):525–562, 1994.
- [LSYZ] Changzheng Li, Ryan M. Shifler, Mingzhi Yang, and Chi Zhang. On frobenius-perron dimension. *arXiv:1909.01693*.
- [Rie01] Konstanze Rietsch. Quantum cohomology rings of Grassmannians and total positivity. *Duke Math. J.*, 110(3):523–553, 2001.

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