SCORE________________

1. Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, evaluate each expression and find their domains:

   (a) $(f+g)(x)$

   [3 points]

   Solution:
   $(f+g)(x) = \sqrt{x} + x^2 - 1, \quad D_{f+g} : [0, \infty)$

   (b) $(f-g)(x)$

   [3 points]

   Solution:
   $(f-g)(x) = \sqrt{x} - x^2 + 1, \quad D_{f-g} : [0, \infty)$

   (c) $(f \cdot g)(x)$

   [3 points]

   Solution:
   $(f \cdot g)(x) = \sqrt{x}(x^2 - 1), \quad D_{f \cdot g} : [0, \infty)$

   (d) $(f/g)(x)$

   [3 points]

   Solution:
   $(f/g)(x) = \dfrac{\sqrt{x}}{x^2 - 1}, \quad D_{f/g} : [0, 1) \cup (1, \infty)$

   (e) $(f \circ g)(x)$

   [3 points]

   Solution:
   $(f \circ g)(x) = \sqrt{x^2 - 1}, \quad D_{f \circ g} : (-\infty, -1] \cup [1, \infty)$

2. Find the inverse of $f(x) = \sqrt{x}$ from the previous problem and find its domain.

[5 points]

Solution:

$f^{-1}(x) = x^2, \quad D_{f^{-1}} : [0, \infty)$ since $R_f : [0, \infty)$.

3. Evaluate the following:

$$\lim_{x \to -1} (2x^3 - 12x - 4)$$

[5 points]

Solution:

$\lim_{x \to -1} (2x^3 - 12x - 4) = 2(-1)^3 - 12(-1) - 4 = 6$

4. Evaluate the following:

$$\lim_{x \to 5} \frac{x^2 - 2x - 15}{x - 5}$$

[5 points]

Solution:

$$\lim_{x \to 5} \frac{x^2 - 2x - 15}{x - 5} = \lim_{x \to 5} \frac{(x-5)(x+3)}{x-5} = \lim_{x \to 5} (x+3) = 5 + 3 = 8$$

5. Given the function:

$$f(x) = \begin{cases} 2 - x & \text{when } x < -1 \\ x & \text{when } -1 \leq x < 1 \\ (x-1)^2 & \text{when } x \geq 1 \end{cases}$$

find the following limits (if they exist).

(a) $\lim_{x \to -1^-} f(x)$

[5 points]

Solution:

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 2 - x = 2 - (-1) = 3$

(b) $\lim_{x \to -1^+} f(x)$

[5 points]

Solution:

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x = -1$

(c) $\lim_{x \to -1} f(x)$

[5 points]

Solution:

$\lim_{x \to -1} f(x)$ DNE

6. Give the formal definition of a limit.

[5 points]

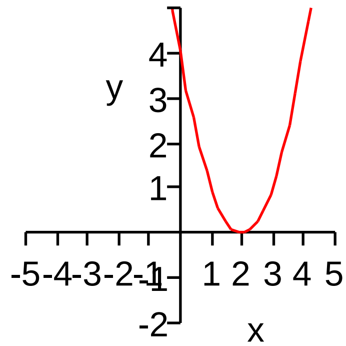
Solution:

$\lim_{x \to a} f(x) = L$,

if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

7. The graph of f(x) is given. Draw the graph of f(x - 2) - 1.

[5 points]



8. Using the limit definition, find the derivative of $f(x) = x^2 + 2x + 3$.

[5 points]

9. Given the function f(x),

x	f(x)
0.5	2.5
0.9	8.1
0.99	9.8
1	10
1.01	10.2
1.1	12.1
1.5	22.5

Estimate the slope of the tangent line at $x = 1$ by calculating the slopes of the secant lines over the following intervals:

$$(0.5, 1), (0.9, 1), (0.99, 1), (1, 1.01), (1, 1.1), (1, 1.5)$$

[9 points]

10. Find an equation of the tangent line to $f(x) = x^4 - x$ at the point (-1, 2).

[10 points]

Solution:

$f'(x) = 4x^3 - 1$

$f'(-1) = -5$

Equation of the tangent line: $y - 2 = -5(x + 1)$

11. Determine the vertical and horizontal aysmptotes (if any exist) of the following function:

$$f(x) = \frac{x^2 + 4}{x^2 - 1}$$

[10 points]

Solution:

Vertical asymptotes: $x = 1$ and $x = -1$

Horizontal asymptotes: $y = 1$

12. Find the derivative of the function $f(x) = 3x(2x + 5)^6$.

[7 points]

Solution:

Product & Chain rule: $f'(x) = 3(2x + 5)^6 + 36x(2x + 5)^5$

13. Find the derivative of the function $f(x) = 5x^4 - \ln(2x^2) + xe^x$.

[5 points]

Solution:

$f'(x) = 20x^3 - \dfrac{4x}{2x^2} + e^x + xe^x = 20x^3 - \dfrac{2}{x} + e^x + xe^x$

14. Find $\dfrac{dy}{dx}$ by implicit differentiation.

$$xe^y - 10x + 3y = 0$$

[5 points]

Solution:

$$\begin{aligned}
\frac{d}{dx}[xe^y - 10x + 3y] &= \frac{d}{dx}[0] \\
e^y + xe^y\frac{dy}{dx} - 10 + 3\frac{dy}{dx} &= 0 \\
xe^y\frac{dy}{dx} + 3\frac{dy}{dx} &= 10 - e^y \\
\frac{dy}{dx}(xe^y + 3) &= 10 - e^y \\
\frac{dy}{dx} &= \frac{10 - e^y}{xe^y + 3}
\end{aligned}$$

15. A companys profit from selling x units of an item is $P = 1000x - \frac{1}{2}x^2$ dollars. If sales are growing at a rate of 20 per day, find how rapidly profit is growing (in dollars per day) when 600 units have been sold.

[5 points]

Solution:

Related rates problem: Given $\dfrac{dx}{dt} = 20$, $x = 600$, we want to find $\dfrac{dP}{dt}$.

$$\begin{aligned}
\frac{d}{dt}[P] &= \frac{d}{dt}\left[1000x - \frac{1}{2}x^2\right] \\
\frac{dP}{dt} &= (1000 - x)\frac{dx}{dt} \\
\frac{dP}{dt} &= (1000 - 600)(20) \\
\frac{dP}{dt} &= 8000
\end{aligned}$$

Profit is growing at a rate of \$8000 per day when 600 units have been sold.

16. Find the absolute extrema of the function on the closed interval $[-1, 2]$.

$$g(x) = 3x^4 - 4x^3$$

[8 points]

Solution:

Find critical numbers: $g'(x) = 12x^3 - 12x^2$

$$\begin{aligned}
g'(x) &= 0 \\
12x^3 - 12x^2 &= 0 \\
12x^2(x - 1) &= 0
\end{aligned}$$

Critical numbers: $x = 0, 1$. Evaluate at endpoints and critical numbers.

$$\begin{aligned}
g(-1) &= 7 \\
g(0) &= 0 \\
g(1) &= -1, \text{ absolute minimum} \\
g(2) &= 16, \text{ absolute maximum}
\end{aligned}$$

17. Find all numbers c that satisfy the conclusion of the Mean Value Theorem on the interval $[0, 2]$

$$f(x) = x^3 + x - 1$$

[6 points]

Solution:

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$ for some $c \in [a, b]$ if $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b).

$$\begin{aligned} f'(x) &= 3x^2 + 1 \\ \frac{f(2) - f(0)}{2 - 0} &= 5 \end{aligned}$$

Find c such that $f'(c) = 5$:

$$\begin{aligned} 3c^2 + 1 &= 5 \\ c &= \pm\sqrt{\tfrac{4}{3}} \end{aligned}$$

$\sqrt{\frac{4}{3}} \in [0, 2]$

18. Let $V(t) = 16e^{-0.5t}$ be the amount of coffee (in ounces) left in Star's cup t minutes after she receives her drink. Find and interpret $V'(2)$.

[6 points]

Solution:

$$\begin{aligned} V'(x) &= -8e^{-0.5t} \\ V'(2) &= -8e^{-0.5(2)} \\ V'(2) &= -8e^{-1} \\ V'(2) &\approx -2.94 \end{aligned}$$

At 2 minutes, Star is drinking the coffee at a rate of approximately 2.94 oz/min.

19. Let $f(x) = -2x^3 + 4x + 3$.

(a) Find all relative extrema.

[5 points]

Solution:
Relative minimum at $x = -\sqrt{\frac{2}{3}}$, (-0.82, -1.35);
Relative maximum at $x = \sqrt{\frac{2}{3}}$, (0.82, 5.18).

(b) Find the open interval(s) on which $f(x)$ is increasing and/or decreasing.

[5 points]

Solution:
Increasing on $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$
Decreasing on $\left(\infty, -\sqrt{\frac{2}{3}}\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$

(c) Find the largest open interval(s) on which $f(x)$ is concave up and those on which $f(x)$ is concave down.

[5 points]

Solution:
Concave downward on $(0, \infty)$
Concave upward on $(-\infty, 0)$

(d) Find all inflection points.

[5 points]

Solution:
Inflection point at $(0, 3)$.

20. Use L'Hopital's Rule to evaluate the following limits. If L'Hôpital's Rule does not apply, explain why.

(a) $\lim_{t \to 0} \frac{e^{3t} - 1}{t}$

[5 points]

Solution:

$$\lim_{t \to 0} \frac{e^{3t} - 1}{t} = \lim_{t \to 0} \frac{3e^{3t}}{1} = 3$$

(b) $\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}}$

[5 points]

Solution:

L'Hôpital's Rule gives us a cycling expression, cannot be applied.

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} \cdot \frac{\frac{1}{x}}{\sqrt{\frac{1}{x^2}}} = \lim_{x \to \infty} \frac{1}{\sqrt{1 + \frac{1}{x^2}}} = 1$$

21. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 180,000 square meters in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is needed along the river?

[10 points]

Solution:

Dimensions: 300×600

22. The average pollen count in New York City on day x of the pollen season is $P(x) = 8x - 0.2x^2$ (for $0 \le x \le 40$). On which day is the pollen count highest?

[5 points]

Solution:

Absolute extrema problem:

Find critical numbers: $P'(x) = 8 - 0.4x \Leftrightarrow P'(x) = 0 \Leftrightarrow 8 - 0.4x = 0 \Rightarrow x = 20$

$$\begin{aligned} P(0) &= 0 \\ P(20) &= 80,\ \text{absolute maximum} \\ P(40) &= 0 \end{aligned}$$

Pollen count is highest on day 20.

23. Find the linearization, that is, the tangent line approximation of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.99}$ and $\sqrt{4.03}$.

[8 points]

Solution:

Tangent line approximation: $y = f(a) + f'(a)(x - a)$

$$f'(x) = \frac{1}{2}(x+3)^{(-1/2)},\ \ f'(1) = \frac{1}{4}$$

Tangent line approximation: $y = f(1) + f'(1)(x - 1) = 2 + \frac{1}{4}(x - 1)$

Let $g(x) = 2 + \frac{1}{4}(x - 1),\ \ g(x) \approx f(x)$ for x near a.

$g(.99) = 2 + \frac{1}{4}(.99 - 1) = 1.9975 \approx f(.99) = \sqrt{3.99}$

$g(1.03) = 2 + \frac{1}{4}(1.03 - 1) = 2.0075 \approx f(1.03 = \sqrt{4.03}$

24. Find the most general antiderivative of the following functions.

(a) $f(x) = 1 - x^3 + 5x^5 - 3x^7$

[5 points]

Solution:
$F(x) = x - \frac{1}{4}x^4 + \frac{5}{6}x^6 + \frac{3}{8}x^8 + C$

(b) $f(x) = 2x + 3x^{1.7}$

[5 points]

Solution:
$F(x) = x^2 + \frac{3}{2.7}x^{2.7} + C$

25. Find the antiderivative f of f' that satisfies the given condition.

(a) $f'(x) = 8x^3 + 12x + 3, f(1) = 6$

[5 points]

Solution:
$f(x) = 2x^4 + 6x^2 + 3x + C$

$f(1) = 6 \implies 2(1)^4 + 6(1)^2 + 3(1) + C = 6 \implies 11 + C = 6$

$C = -5 \implies f(x) = 2x^4 + 6x^2 + 3x - 5$

(b) $f'(x) = \sqrt{x}(6 + 5x), f(1) = 10$

[5 points]

Solution:
$f'(x) = 6\sqrt{x} + 5x\sqrt{x}) = 6x^{1/2} + 5x^{3/2}$

$f(x) = 4x^{3/2} + 2x^{5/2} + C$

$f(1) = 10 \implies 4(1)^{3/2} + 2(1)^{5/2} + C = 10 \implies 6 + C = 10$

$C = 4 \implies f(x) = 4x^{3/2} + 2x^{5/2} + 4$