SCORE______________

1. Use the Squeeze theorem to find the $\lim_{x \to 0} f(x)$ if

$$3 - x^3 \leq f(x) \leq 3 + x^3.$$

[5 points]

Solution:

$$\begin{aligned} \text{Now, } \lim_{x \to 0} 3 - x^3 &= 3 \\ \& \lim_{x \to 0} 3 + x^3 &= 3 \\ \text{Therefore, } \lim_{x \to 0} f(x) &= 3 \end{aligned}$$

2. Evaluate the following

$$\lim_{x \to 2} \frac{x+2}{x^2+3x+5}.$$

[5 points]

3. Give the formal definition of a limit.

[5 points]

4. Find a suitable δ which proves that

$$\lim_{x \to -3} (9 - 2x) = 15$$

[6 points]

Solution:

$$\begin{aligned}
|(9-2x)-15| &\leq k|x+3| \\
\text{Work from the left-hand side} & \\
|(9-2x)-15| &= |-6-2x| \\
&= |(-2)(x+3)| \\
&= |(-2)||(x+3)| \\
&= 2|x+3| \\
\text{Thus, k} = 2 & \\
\text{Now } \delta = \epsilon/k, \text{ so } \delta = \epsilon/2. &
\end{aligned}$$

5. Find the following limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - a| < \delta$.

$$\lim_{x \to 2} (2x + 8)$$

[6 points]

6. Find the x-values (if any) at which f is not continuous.

$$f(x) = \frac{x+1}{x^2 - 4x + 3}$$

[8 points]

7. Use the Intermediate Value Theorem to show that there is a c, $a \leq c \leq b$, such that $f(c) = 9$ on the interval $[0, 5]$.

$$f(x) = x^2 + x - 1$$

[8 points]

8. Evaluate the following limit

$$\lim_{x \to \infty} \frac{\sqrt{4x^2 - 2}}{3x + 1}.$$

[8 points]

9. Determine the vertical and horizontal asymptotes (if any exist) of the following function

$$f(x) = \frac{3x-15}{x^2-25}.$$

[10 points]

Solution:

$$\begin{aligned}
\text{Vertical Asymptotes:} & \\
x^2 - 25 &= 0 \\
(x+5)(x-5) &= 0 \\
x &= \pm 5
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 5} \frac{3x-15}{x^2-25} &= \lim_{x\to 5} \frac{3(x-5)}{(x-5)(x+5)} \\
&= \lim_{x\to 5} \frac{3}{(x+5)} \\
&= \tfrac{3}{10}
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to -5^+} \frac{3x-15}{x^2-25} &= \lim_{x\to -5^+} \frac{3(x-5)}{(x-5)(x+5)} \\
&= \lim_{x\to -5^+} \frac{3}{(x+5)} \\
&= \infty \\
\text{Vertical Asymptote: x = -5} &
\end{aligned}$$

$$\begin{aligned}
\text{Horizontal Asymptotes:} & \\
\lim_{x\to \infty} \frac{3x-15}{x^2-25} &= \lim_{x\to \infty} \frac{3x-15}{x^2-25} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= \lim_{x\to \infty} \frac{\frac{3}{x} - \frac{15}{x^2}}{1 - \frac{25}{x^2}} \\
&= \tfrac{0-0}{1-0} \\
&= 0 \\
\text{Horizontal Asymptote: y = 0} &
\end{aligned}$$

$$\text{Note: } \lim_{x\to -\infty} \frac{3x-15}{x^2-25} = 0 \text{ also.}$$

10. Using the limit definition, find the derivative of $f(x) = x^2 + x - 1$.

[8 points]

11. Using the limit definition, find the derivative of $f(x) = \dfrac{2}{x+1}$.

[8 points]

12. Find the derivative of $f(x) = 2x^4 - 3x^2 + \pi x - 18$.

[7 points]

Solution:

$$f'(x) = 2(4)x^{4-1} - 3(2)x^{2-1} + \pi x^{1-1} - 0$$

$$f'(x) = 8x^3 - 6x + \pi$$

13. Find the derivative of $f(x) = 6x^5 + \frac{8}{x^2} - 3e^x$

[7 points]

Solution:

$$f'(x) = \frac{d}{dx}[6x^5] + \frac{d}{dx}[8x^{-2}] - \frac{d}{dx}[3e^x]$$

$$f'(x) = 6(5)x^4 + 8(-2)x^{-3} - 3e^x = 30x^4 - 16x^{-3} - 3e^x$$

Bonus. Do only one of the

(a) Prove that $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$, (b) Find a suitable δ which proves $\lim_{x\to a} \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{a}}$

[6 points]