SCORE_______________

1. Find functions f and g such that the given function is $f(g(x))$ and find the derivative.

$$(2x^2 - 7x + 6)^9$$

[7 points]

2. Find the derivative of the function $f(x) = \sin 2x(3x+7)^5$.

[7 points]

Solution:

$$\begin{aligned} f'(x) &= \tfrac{d}{dx}[\sin 2x](3x+7)^5 + \tfrac{d}{dx}[(3x+7)^5](\sin 2x) \\ &= \cos 2x(2)(3x+7)^5 + 5(3x+7)^4(3)(\sin 2x) \\ &= 2\cos 2x(3x+7)^5 + 15\sin 2x(3x+7)^4 \end{aligned}$$

3. Find $f'(5)$, if $f(x) = x^2 e^{0.2x}$

[8 points]

4. Find an equation of the tangent line to the curve at the point (0,0).

$$y = \sin x + \sin^2 x$$

[8 points]

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}[\sin x + \sin^2 x] \\
&= \frac{d}{dx}[\sin x] + \frac{d}{dx}[\sin^2 x] \\
&= \cos x + \frac{d}{dx}[(\sin x)^2] \\
&= \cos x + 2\sin x \cos x \\
\left.\frac{dy}{dx}\right|_{(0,0)} &= \cos 0 + 2\sin 0 \cos 0 \\
&= 1 + 2(0)(1) = 1 \\
\text{tangent line slope} &\text{ is } 1. \\
y - 0 &= 1(x - 0) \\
y &= x
\end{aligned}$$

5. Find $f''(x)$, given the following

$$f(x) = x\cos x$$

[8 points]

Solution:

$$\begin{aligned}
f'(x) &= \tfrac{d}{dx}[x]\cos x + \tfrac{d}{dx}[\cos x](x) \\
&= \cos x + (-\sin x)(x) \\
&= \cos x - x\sin x \\
f''(x) &= \tfrac{d}{dx}[\cos x] + \tfrac{d}{dx}[x\sin x] \\
&= -\sin x + \tfrac{d}{dx}[x]\sin x + \tfrac{d}{dx}[\sin x](x) \\
&= -\sin x + \sin x + \cos x(x) \\
&= -\sin x + \sin x + x\cos x \\
&= x\cos x
\end{aligned}$$

6. Suppose that $f(2) = -3, g(2) = 2, f'(2) = -2,$ and $g'(2) = 1$. Find $h'(2)$

$$h(x) = [f(x)g(x)]^2$$

[8 points]

7. Two ships leave the same port at noon. Ship A sails north at 5 mph, and ship B sails east at 12 mph. How fast is the distance between them changing at 1 p.m.?

[8 points]

8. Find y' using implicit differentiaion

$$2y^2 - 3xy - x = 6$$

[8 points]

9. Find $\dfrac{dy}{dx}$, given $y = (\sin x)^{\cos x}$ (Hint: Use logarithmic differentiation).

[8 points]

Solution:

$$\begin{aligned}
\ln y &= \ln\left((\sin x)^{\cos x}\right) \\
&= \cos x \ln(\sin x) \\
\tfrac{d}{dx}[\ln y] &= \tfrac{d}{dx}[\cos x \ln(\sin x)] \\
\tfrac{1}{y}\tfrac{dy}{dx} &= \tfrac{d}{dx}[\cos x](\ln(\sin x)) + \tfrac{d}{dx}[\ln \sin x](\cos x) \\
&= -\sin x(\ln(\sin x)) + \tfrac{1}{\sin x}(\cos x)(\cos x) \\
&= -\sin x \ln(\sin x) + \cot x \cos x \\
\tfrac{1}{y}\tfrac{dy}{dx} &= -\sin x \ln(\sin x) + \cot x \cos x \\
\tfrac{dy}{dx} &= y(-\sin x \ln(\sin x) + \cot x \cos x) \\
\tfrac{dy}{dx} &= (\sin x)^{\cos x}(-\sin x \ln(\sin x) + \cot x \cos x)
\end{aligned}$$

10. Use differentials (or equivalently, a linear approximation) to approximate $\sqrt[3]{7.8}$.

[8 points]

11. Find the absolute maximum and minimum of the function on the closed interval $[-3, 1]$

$$f(x) = x^3 + 3x^2 - 1$$

[7 points]

12. The average speed of a vehicle on a stretch of Route 134 between 6 a.m. and 10 a.m. on a typical weekday is approximated by the function

$$f(t) = 20t - 40\sqrt{t} + 50, \qquad (0 \le t \le 4)$$

where $f(t)$ is measured in miles per hour and t is measured in hours, with $t = 0$ corresponding to 6 a.m. At what time of the morning commute is the traffic moving at the slowest rate? What is the average speed of a vehicle at that time?

[10 points]

This is an absolute extrema problem.

Solution:

$$\begin{aligned} f'(t) &= 20 - 40(\tfrac{1}{2})t^{-1/2} \\ &= 20 - 20t^{-1/2} \\ &= 20 - \frac{20}{\sqrt{t}} \end{aligned}$$

Find critical numbers: f is undefined at $t = 0$

$$\begin{aligned} 0 &= f'(t) \\ 0 &= 20 - \frac{20}{\sqrt{t}} \\ \frac{20}{\sqrt{t}} &= 20 \\ 20 &= 20\sqrt{t} \\ 1 &= \sqrt{t} \\ 1 &= t \end{aligned}$$

critical numbers: $t = 0,\ t = 1$

$f(0) = 50,\ f(1) = 30 \qquad f(4) = 50$

Traffic is slowest at $t = 1$ which corresponds to 7 a.m. and the average speed at that time is 30 mph.

13. Sketch the graph of $f(x) = x^2$ on the interval $0 \leq x \leq 2$, use your sketch to determine the absolute and relative maximum and minimum values of f.

[5 points]

Bonus. Use implicit differention to find the following derivative:

$$\frac{d}{dx}(\sec^{-1} x)$$

[6 points]