SCORE______________

1. Find functions f and g such that the given function is $f(g(x))$ and find the derivative.

$$(2x^2 - 7x + 6)^9$$

[7 points]

2. Find the derivative of the function $f(x) = \sin 2x(3x+7)^5$.

[7 points]

Solution:

$$\begin{aligned} f'(x) &= \tfrac{d}{dx}[\sin 2x](3x+7)^5 + \tfrac{d}{dx}[(3x+7)^5](\sin 2x) \\ &= \cos 2x(2)(3x+7)^5 + 5(3x+7)^4(3)(\sin 2x) \\ &= 2\cos 2x(3x+7)^5 + 15\sin 2x(3x+7)^4 \end{aligned}$$

3. Find $f'(5)$, if $f(x) = x^2 e^{0.2x}$

[8 points]

Solution:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[x^2 e^{0.2x}] \\
&= \frac{d}{dx}[x^2](e^{0.2x}) + \frac{d}{dx}[e^{0.2x}]x^2 \\
&= 2xe^{0.2x} + 0.2e^{0.2x}x^2 \\
f'(5) &= 2(5)e^{0.2(5)} + 0.2e^{0.2(5)}5^2 \\
&= 10e + 5e \\
&= 15e \approx 40.77
\end{aligned}$$

4. Find an equation of the tangent line to the curve at the point (0,0).

$$y = \sin x + \sin^2 x$$

[8 points]

Solution:

$$\begin{aligned}
\tfrac{dy}{dx} &= \tfrac{d}{dx}[\sin x + \sin^2 x] \\
&= \tfrac{d}{dx}[\sin x] + \tfrac{d}{dx}[\sin^2 x] \\
&= \cos x + \tfrac{d}{dx}[(\sin x)^2] \\
&= \cos x + 2\sin x \cos x \\
\tfrac{dy}{dx}\Big|_{(0,0)} &= \cos 0 + 2\sin 0 \cos 0 \\
&= 1 + 2(0)(1) = 1 \\
\text{tangent line slope} &\ \text{is}\ \ 1. \\
y - 0 &= 1(x - 0) \\
y &= x
\end{aligned}$$

5. Find $f''(x)$, given the following

$$f(x) = x\cos x$$

[8 points]

Solution:

$$\begin{aligned} f'(x) &= \tfrac{d}{dx}[x]\cos x + \tfrac{d}{dx}[\cos x](x) \\ &= \cos x + (-\sin x)(x) \\ &= \cos x - x\sin x \\ f''(x) &= \tfrac{d}{dx}[\cos x] + \tfrac{d}{dx}[x\sin x] \\ &= -\sin x + \tfrac{d}{dx}[x]\sin x + \tfrac{d}{dx}[\sin x](x) \\ &= -\sin x + \sin x + \cos x(x) \\ &= -\sin x + \sin x + x\cos x \\ &= x\cos x \end{aligned}$$

6. Suppose that $f(2) = -3, g(2) = 2, f'(2) = -2,$ and $g'(2) = 1$. Find $h'(2)$

$$h(x) = [f(x)g(x)]^2$$

[8 points]

Solution:

$$\begin{aligned} h'(x) &= \frac{d}{dx}[[f(x)g(x)]^2] \\ &= 2[f(x)g(x)] \cdot \frac{d}{dx}[f(x)g(x)] \\ &= 2[f(x)g(x)] \cdot (f'(x)g(x) + g'(x)f(x)) \\ h'(2) &= 2[f(2)g(2)] \cdot (f'(2)g(2) + g'(2)f(2)) \\ &= 2(-3)(2)((-2)(2) + (1)(-3)) \\ &= (-12)(-4-3) = 84 \end{aligned}$$

7. Two ships leave the same port at noon. Ship A sails north at 5 mph, and ship B sails east at 12 mph. How fast is the distance between them changing at 1 p.m.?

[8 points]

8. Find y' using implicit differentiaion

$$2y^2 - 3xy - x = 6$$

[8 points]

Solution:

$$\begin{aligned}
\frac{d}{dx}[2y^2 - 3xy - x] &= \frac{d}{dx}[6] \\
4y\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} - 1 &= 0 \\
(4y - 3x)\frac{dy}{dx} - 3y - 1 &= 0 \\
(4y - 3x)\frac{dy}{dx} &= 3y + 1 \\
\frac{dy}{dx} &= \frac{3y + 1}{4y - 3x}
\end{aligned}$$

9. Find $\dfrac{dy}{dx}$, given $y = (\sin x)^{\cos x}$ (Hint: Use logarithmic differentiation).

[8 points]

Solution:

$$\begin{aligned}
\ln y &= \ln\left((\sin x)^{\cos x}\right) \\
&= \cos x \ln(\sin x) \\
\tfrac{d}{dx}[\ln y] &= \tfrac{d}{dx}[\cos x \ln(\sin x)] \\
\tfrac{1}{y}\tfrac{dy}{dx} &= \tfrac{d}{dx}[\cos x](\ln(\sin x)) + \tfrac{d}{dx}[\ln \sin x](\cos x) \\
&= -\sin x(\ln(\sin x)) + \tfrac{1}{\sin x}(\cos x)(\cos x) \\
&= -\sin x \ln(\sin x) + \cot x \cos x \\
\tfrac{1}{y}\tfrac{dy}{dx} &= -\sin x \ln(\sin x) + \cot x \cos x \\
\tfrac{dy}{dx} &= y(-\sin x \ln(\sin x) + \cot x \cos x) \\
\tfrac{dy}{dx} &= (\sin x)^{\cos x}(-\sin x \ln(\sin x) + \cot x \cos x)
\end{aligned}$$

10. Use differentials (or equivalently, a linear approximation) to approximate $\sqrt[3]{7.8}$.

[8 points]

Solution:

$f(x) = \sqrt[3]{x} = x^{1/3}$, $dx = -0.2$, x $= 8$

$$\begin{aligned}
dy &= f'(x)dx \\
dy &= \frac{1}{3}x^{-2/3}dx \\
dy &= \frac{1}{3}8^{-2/3}(-0.2) \\
dy &= \frac{1}{12}(-0.2) \approx -0.01667 \\
f(x+\Delta x) = f(7.8) &\approx f(8) + dy = 1.98333 \\
\sqrt[3]{7.8} &\approx 1.98333
\end{aligned}$$

Bonus. Use implicit differention to find the following derivative:

$$\frac{d}{dx}(\sec^{-1} x)$$

[6 points]