3. Write the following statement in natural language and then negate it:
$\forall x \in \mathbb{R}, \exists n \in \mathbb{N} \ni n > x$.

[5 points]

Solution:

For all real numbers x, there exists a natural number n, such that $n > x$.

Negation: $\exists x \in \mathbb{R} \ni \forall n \in \mathbb{N}, n \leq x$.

4. Write the converse and contrapositive to the following statements:

(a) If you are 18, you can vote.

[2 points]

Converse: If you can vote, then you are 18.
Contrapositive: If you cannot vote, then you are not 18.

(b) If I live in Ocean City, then I live in Maryland.

[2 points]

Converse: If I live in Maryland, then I live in Ocean City.
Contrapositive: If I do not live in Maryland, then I do not live in Ocean City.

(c) A square is a four-sided figure.

[2 points]

Converse: A four-sided figure is a square.
Contrapositive: If a figure is not four-sided, then it is not a square.

(d) If it is raining, then there are dark clouds in the sky.

[2 points]

Converse: If there are dark clouds in the sky, then it is raining.
Contrapositive: If there are no dark clouds in the sky, then it is not raining.

(e) If n is an odd integer, then $n^2 + n - 2$ is an even integer.

[2 points]

Converse: If $n^2 + n - 2$ is an even integer, then n is an odd integer.
Contrapositive: If $n^2 + n - 2$ is an odd integer, then n is an even integer.

5. Prove that the product of any two odd numbers is odd.

[10 points]

Solution:

Let x and y be odd numbers, then $x = 2k + 1$ and $y = 2m + 1$ for some integers k, m.

$$\begin{aligned} x \cdot y &= (2k+1)(2m+1) \\ &= 4km + 2k + 2m + 1 \\ &= 2(2km + k + m) + 1 \\ &= 2p + 1 \end{aligned}$$

So, $x \cdot y$ is odd since p is an integer.

6. Show that the statement "All primes are odd" is false.

[5 points]

Solution:

A prime is a number divisible by itself and 1.
2 is prime and is not odd.

7. Use induction to prove that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.

[10 points]

Statement $P(n)$: $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

Basis Step: Show $P(0)$ is true. (Note: Enumeration begins at 0)

LHS: 1
RHS: $2^{0+1} - 1 = 2 - 1 = 1$

These are equal so $P(0)$ is true.

Inductive Step: Show $P(k) \to P(k+1)$.

Induction Hypothesis (IH): Assume $P(k)$ is true, that is,
$P(k)$: $1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= 2^{k+1} - 1 + 2^{k+1} \\ &= 2(2^{k+1}) - 1 \\ &= 2^{(k+1)+1} - 1 \end{aligned}$$

Therefore, $P(k+1)$ is true. Thus, $P(n)$ is true for all nonnegative integers n.

8. Use strong induction to prove that every amount of postage of 8 cents or more can be formed using just 3-cent and 5-cent stamps.

[10 points]

Statement: Postage of 8 cents or more can be formed by 3- and 5-cent stamps.

Basis Step: Show true for $n = 8, n = 9, n = 10$.

$n = 8$: (8-cent postage) one 3-cent stamp, one 5-cent stamp
$n = 9$: (9-cent postage) three 3-cent stamps
$n = 10$: (10-cent postage) two 5-cent stamps

We've shown statement is true for $n = 8, n = 9, n = 10$.

Inductive Step: Show if the statement is true for $8 \leq j \leq k$, then it is true for $k + 1$.

Induction Hypothesis (IH): Assume statement is true for $8 \leq j \leq k$, $k \geq 10$.

By IH, we know that statement is true for $k - 2$, that is, postage of $k - 2$ can be formed by 3- and 5-cent stamps.We can represent this by $k - 2 = 3m + 5n$, where m and n are integers. Now $k+1 = (k-2)+3 = 3m+5n+3 = 3(m+1)+5n$, so postage of $k + 1$ can be formed by 3- and 5-cent stamps. Therefore, the statement is true for natural numbers greater than 8.

9. How many bit strings of length 10 begin and end with 0? *(Remember bit strings are made up of 0's and 1's only).*

[6 points]

Solution:

Ordered sample with repetition: 2^8

10. (a) Compute $\binom{6}{4}$

[6 points]

Solution:

$$\begin{aligned}
\binom{6}{4} &= \frac{6!}{(6-4)!4!} \\
&= \frac{6!}{(2)!4!} \\
&= \frac{6 \cdot 5 \cdot 4!}{(2)!4!} \\
&= \frac{6 \cdot 5}{2} \\
&= 15
\end{aligned}$$

(b) Find $(x+1)^4$ using binomial coefficients.

[6 points]

Solution:

$$\begin{aligned}
(x+1)^4 &= \binom{4}{0}x^4 + \binom{4}{1}x^3 \cdot 1 + \binom{4}{2}x^2 \cdot 1^2 + \binom{4}{3}x \cdot 1^3 + \binom{4}{4}1^4 \\
&= x^4 + 4x^3 + 6x^2 + 4x + 1
\end{aligned}$$

11. A club has twenty members. In how many ways can they chose a slate of four officers consisting of a president, vice-president, secretary, and treasurer?

[6 points]

Solution:

Ordered sample without repetition:

$$\begin{aligned} P(20,4) &= \frac{20!}{(20-4)!} \\ &= \frac{20!}{16!} \\ &= 20 \cdot 19 \cdot 18 \cdot 17 \\ &= 116280 \end{aligned}$$

12. A group of people is comprised of six from Pennsylvania, seven from Vermont, and eight from Illinois. In how many ways can a committee of six be formed with two people from each state?

[6 points]

Solution:

Unordered sample without repetition:

$$\begin{aligned} \binom{6}{2} \cdot \binom{7}{2} \cdot \binom{8}{2} &= 15 \cdot 21 \cdot 28 \\ &= 8820 \end{aligned}$$