SCORE_______________

1. A 2-kg mass is attached to a spring with stiffness $k = 50$ N/m. The mass is displaced 1/4 m to the left of the equilibrium point and given a velocity of 1 m/sec to the left. Neglecting damping, find the equation of motion of the mass along with the amplitude, period and frequency. How long after release does the mass pass through the equilibrium position?

[12 points]

Differential Equation: $2y'' + 50y = 0, \;\; y(0) = -1/4, \;\; y'(0) = -1$

Equation of motion: $y(t) = -\dfrac{1}{4}\cos 5t - \dfrac{1}{5}\sin 5t$

Time: $t \approx .45$ sec

2. A 32-lb weight is attached to a vertical spring, causing it to stretch 6 in. upon coming to rest at equilibrium. The damping constant for the system is 2 lb-sec/ft. An external force $F(t) = 4\cos 8t$ lb is applied to the weight. Find the steady-state solution for the system.

[12 points]

Differential Equation: $y'' + 2y' + 64y = 4\cos 8t$

Steady-state solution (particular solution): $y_p(t) = \dfrac{1}{4}\sin 8t$

3. Use the elimination method to find a general solution to the given linear system.

$$x' = x - y$$

$$y' = y - 4x$$

[12 points]

Solution:

$x(t) = c_1 e^{-t} + c_2 e^{3t}$

$y(t) = 2c_1 e^{-t} - 2c_2 e^{3t}$

4. Solve the following system of differential equations.

$$x' = 3x - y - 12$$

$$y' = x + y + 4e^t$$

[12 points]

(Hint: Use determinants)
Solution:

$x(t) = c_1 e^{2t} + c_2 t e^{2t} + 3 - 4e^t$

$y(t) = (c_1 - c_2)e^{2t} + c_2 t e^{2t} - 3 - 8e^t$

5. Solve the following differential equation.

$$y''' - 4y'' + y' + 6y = 0$$

[12 points]

Solution: $y(x) = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{2x}$

6. Solve the differential equation

$$\frac{dy}{dx} + \frac{2y}{x} = 2x^2 y^2$$

[12 points]

Solution: $y(x) = (cx^2 - 2x^3)^{-1}$, $y(x) \equiv 0$.

7. Solve the following system initial value problem.

$$y''' + y'' = \cos(2t), \;\; y(0) = 1, \; y'(0) = 2, \; y''(0) = 3$$

[12 points]

Solution: $y(t) = -\frac{7}{4} + 5t + \frac{14}{5}e^{-t} - \frac{1}{20}\cos 2t - \frac{1}{10}\sin 2t$

8. Solve the following differential equation using variation of parameters.

$$y''' + y'' = \sec t$$

[12 points]

Solution:

$$y_p t) = \left(-\int t\sec t\ dt - \ln|\sec t + \tan t|\right) + -t\ln|\sec t + \tan t| + e^{-t}\left(\int e^{t}\sec t\ dt\right)$$

$$y(t) = c_1 + c_2 t + c_3 e^{-t} + y_p(t)$$

Note: There are no elementary functions that are $\int t\sec t\ dt$ and $\int e^t \sec t\ dt$.

9. Find a series solution for the following initial value problem:

$$y'' + xy' + y = 0,\ y(0) = 0,\ y'(0) = 1$$

[12 points]

Form: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

Solution: $y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \dots \cdot (2n+1)}$

10. Both $x = 0$ and $x = -1$ are singular points of the differential equation

$$x^2(x+1)^2y'' + (x^2-1)y' + 2y = 0$$

Classify each as regular or irregular.

[12 points]

Both points are irregular.

12. Solve the following Cauchy-Euler equation

$$x^2y'' + 2xy' - 4y = 0$$

[12 points]

Solution: $y(x) = c_1x^{-1}\cos\left(\frac{15}{2}\ln t\right) + c_2x^{-1}\sin\left(\frac{15}{2}\ln t\right)$

13. Find a differential operator annihilates the given functions: a)$1 - 5x^2 + 8x^3$; b) e^{-3x}

[12 points]

Solution: $a)\ D^4$; $b)\ D + 3$

14. Find a lower bound for the radius of convergence of the series solutions about $x = -1$ for the differential equation

$$(x^2 + 1)y'' + xy' - y = 0$$

[6 points]

Singular points: $\pm i$; lower bound: $\sqrt{2}$

Bonus. Find a series solution about the regular singular point $x = 0$ of

$$x^2 y''(x) - xy'(x) + (1 - x)y(x) = 0, \ x > 0$$

using the method of Frobenius.

[20 points]

Form: $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$

Solution: $y(x) = \sum_{n=0}^{\infty} \frac{a_0}{(n!)^2} x^{n+1}$